P3 -group 7
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Newton's Power Series for Sine & Cosine

Obtained by the means of geometry, integration, and power series inversion

Contents

Introduction

In 1711, Isaac Newton (1642-1727) published a derivation of the power series for the sine in his seminal *Of Analysis by means of Equations with an infinite number of terms* (Newton, 1745a, pp. 335-339). This is the first appearance of the power series for the sine in a European Manuscript (Dunham, 1991, p. 11).

We provide a line-by-line analysis of the relevant sections (§37-47), reconstructing Newton's logic and filling in a few gaps in Newton's derivation, where appropriate. We follow D.T. Whiteside (Whiteside, 1968) in interpreting Fluents as time-varying quantities $y(t)$, Fluxions as their finite time-derivatives $\dot{y}$, and Moments as the instantaneous synchronal increments (historically represented by Newton as $o\dot{z}$, where o is an infinitely small interval of time).

37. How Fluents are generated from their Moments

Newton begins his investigation of the power series for the sine and cosine by reminding the reader that knowing the Moments of a time-varying quantity (or 'Fluent') at all times allows you to calculate that quantity at all times as well. In function notation, knowing $\dot{f}$ at all times allows you to find $f(t)$ by integration.

> "37. Let ABD be any Curve, and AHKB a Rectangle, whoſe Side AH or BK is Unity: And imagine the Right Line DBK to move uniformly from AH, ſo as to deſcribe the Areas ABD and AK; and that BK (1) is the Moment with which AK (x), and BD (y) the Moment with which ABD is gradually encreaſed;

$AHKB$ is a two-dimensional x-axis whose side length is 1 (see Figure 1).

If we make the time-dependence of Newton's fluxional method explicit, we can satisfy Newton's requirement for 'uniform' motion by letting:

$$x = t \tag{1}$$

Then, $x = t \Rightarrow \dot{x} = 1$, and thus the Fluxion of x is 1 and the Moment BK is $1dt$ (or $1dx$). Indeed, Newton himself mentions, in a section of the *Treatise on the Quadrature of Curves* (Newton, 1745b) that that it is useful to take a quantity like x, set its first fluxion equal to 1 and all higher fluxions to 0. This implies $dx = dt$, so time-derivatives become x-derivatives. This is necessary to apply a time-based calculus to questions of pure geometry:

> [...] it is proper to consider some Quantity as flowing uniformly, and for its first fluxion to write Unity, for the subsequent ones, nothing.
>
> — Isaac Newton, (Newton, 1745b, §20, p. 7)

Thus, the "1" or the height of the two-dimensional rectangular x-axis serves a dual role: First, it allows us to compare areas with areas and remain geometrically consistent (Newton also does this in his determination of π (Newton, 1736, p. 86). Second, it is a manifestation of setting $dx = dt$, allowing us to tackle questions of geometry using a time-based calculus.

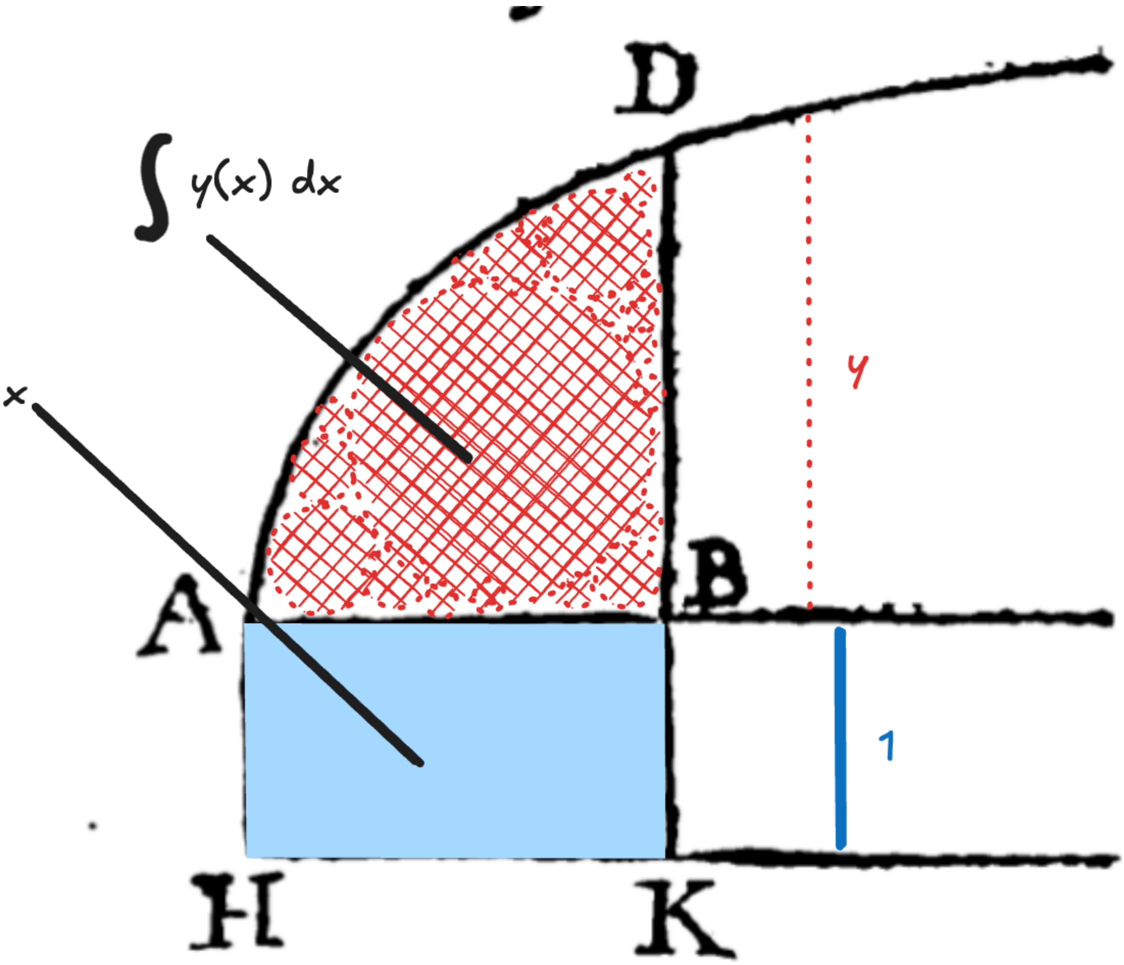
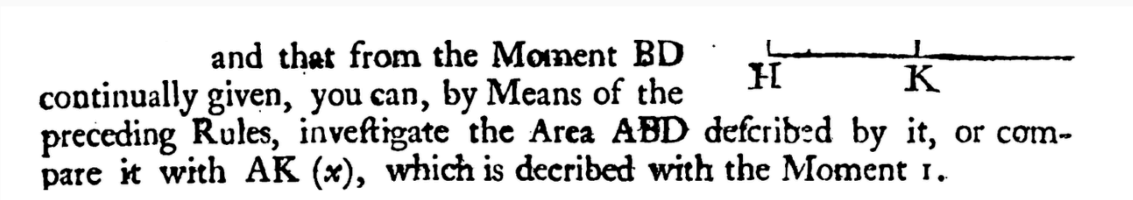


Figure 1: The blue area (x) is "increased continually by the Moment 1". The red area is increased continually by the "Moment" y.

Newton continues:

> 
> and that from the Moment BD continually given, you can, by Means of the preceding Rules, inveftigate the Area ABD defcribed by it, or compare it with AK (x), which is decribed with the Moment 1.

It is puzzling to us that Newton calls BD and AK moments rather than fluxions, because we would recognise $y(x)$ and $\dot{x} = 1$ as derivatives of the blue and red (see Figure 1) areas respectively, rather than infinitesimal area segments.

In Newton's discussion of the Moment in the *Quadrature Of Curves*, he clearly views them as infinitesimal line segments (proportional to an infinitesimal quantity o):

> For let o be a very small Quantity, and let $o\dot{z}, o\dot{y}, o\dot{x}$ be the Moments, that is the momentaneous synchronal Increments of the Quantities z, y, x. And if the flowing

Quantities are just now z, y, x, then after a Moment of Time, being increas'd by their Increments $o\dot{z}, o\dot{y}, o\dot{x}$, these Quantities shall become $z + o\dot{z}, y + o\dot{y}, x + o\dot{x}$

— (Newton, 1745b, §17, p. 6)

Newton's usage of 'Moment' must have changed between the *Quadrature of Curves* (Newton, 1745b) and the *Analysis* (Newton, 1745a), as he writes 1 for the Moment here (rather than instead of $1o$).

One explanation for why Newton avoids the explicit mention of infinitesimal quantities in his treatment is that Newton wants to avoid being seen to apply the Leibnizian second postulate[1] of calculus, which he explicitly rejects in the opening of the *Analysis*:

I consider mathematical Quantities in this Place **not as consisting of very small Parts**; but as describ'd by a continued Motion. Lines are describ'd, and thereby generated not by the Apposition of Parts, but by the continued Motion of Points; Superficies's by the Motion of Lines; Solids by the Motion of Superficies'; Angles by the Rotation of the Sides; Portions of Time by a continual Flux: and so in other Quantities.

— (Newton, 1745a, p. 1, §1), emphasis ours

Perhaps if he had written $1o$ and $\dot{y}o$ for the Moments instead of 1 and $\dot{y}$, he would have left himself open to the accusation that mathematical Quantities *do* "consist of very small Parts".

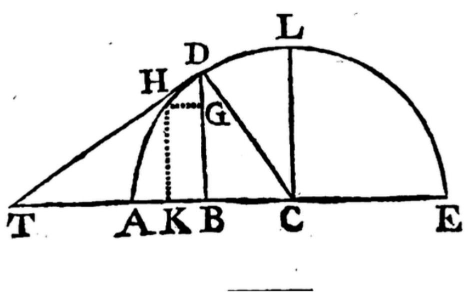
And indeed, since Newton's arguments depend only on ratios between moments, the common factor o may be suppressed without affecting the resulting relations.

38. Finding the arc length of a circle in terms of a coordinate along its diameter.

In this section, Newton uses his method of *inverse fluxions* (integration) to find an arbitrary arc length of a circle in terms of a coordinate x along its diameter (see Figure 2). He first finds an expression of the tangent to the circle as a function of x as a power series, and then integrates it term-by-term to find the arc length.

"

38. Let ADLE be a Circle, the Length of whose Arch AD is to be investigated. Draw the Tangent DHT, and having compleated the indefinitely small Rectangle HGBK, and put AE = 1 = 2AC, it shall be as BK or GH the Moment of the Base AB (x) to HD the Moment of the Arch AD : : BT : DT : : BD ($\sqrt{x-xx}$) : DC ($\frac{1}{2}$) : : 1 (BK) : $\frac{1}{2\sqrt{x-xx}}$ (DH).



[1]Any Curved line consists of infinitely many straight lines, each of which is infinitely small (Bradley et al., 2015)

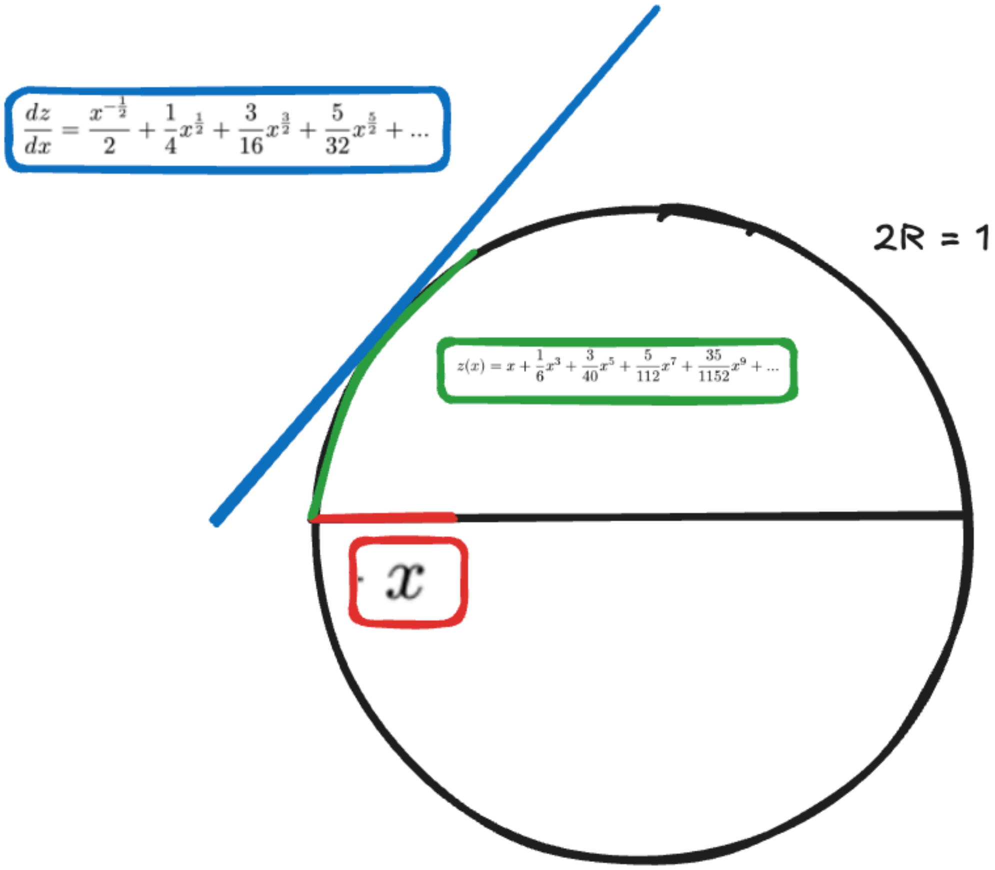


Figure 2: Summary of §38, showing the coordinate along the diameter x, the series expansion of the arc length z, and the tangent to the arc $\frac{dz}{dx}$

Interestingly, whereas §38 is presented as concrete example of §37, the dimensionalities are different. Whereas §37 is about the integral of the line $y(x)$ into an area, §38 is about the integral of the line segments making up the arc to find the total arc length. These are different conceptually. In the former case, integration increases the dimensionality of the object being integrated, whereas in the latter it does not. It is strange that one is given as an example of the other.

The double colon "::" in this passage is just an equals sign. When Newton writes "BD $(\sqrt{x - xx})$: DC $\left(\frac{1}{2}\right)$", the brackets should not be taken to mean multiplication. Instead, they mean something akin to "in other words". So, the statement, "$A(B) : C(D)$" should be read as "A(in other words, B) : C(in other words, D)", or, equivalently $\frac{A}{C} = \frac{B}{D}$. So, the final ratio "$1(BK) : \frac{1}{2\sqrt{x-x^2}}(DH)$" actually means "$\frac{DH}{BK} = \frac{1}{2\sqrt{x-x^2}}$".

We have taken the liberty of rewriting Newton's terse one-liner in modern notation:

$$\frac{dz}{dx} \equiv \frac{DH}{GH} = \frac{DT}{BT} = \frac{DC}{DB} = \frac{1}{2\sqrt{x - x^2}} = \frac{DH}{BK} \tag{2}$$

Where does this expression come from? We start by noticing that the red and green triangles are similar (see Figure 3) since the two triangles share the angle $\angle BDT$ (or $\angle GDH$) and both triangles contain a right angle ($\angle TBD$ and $\angle HGD$). So, by similarity:

$$\frac{DH}{GH} = \frac{DT}{BT} \equiv \frac{dz}{dx} \tag{3}$$

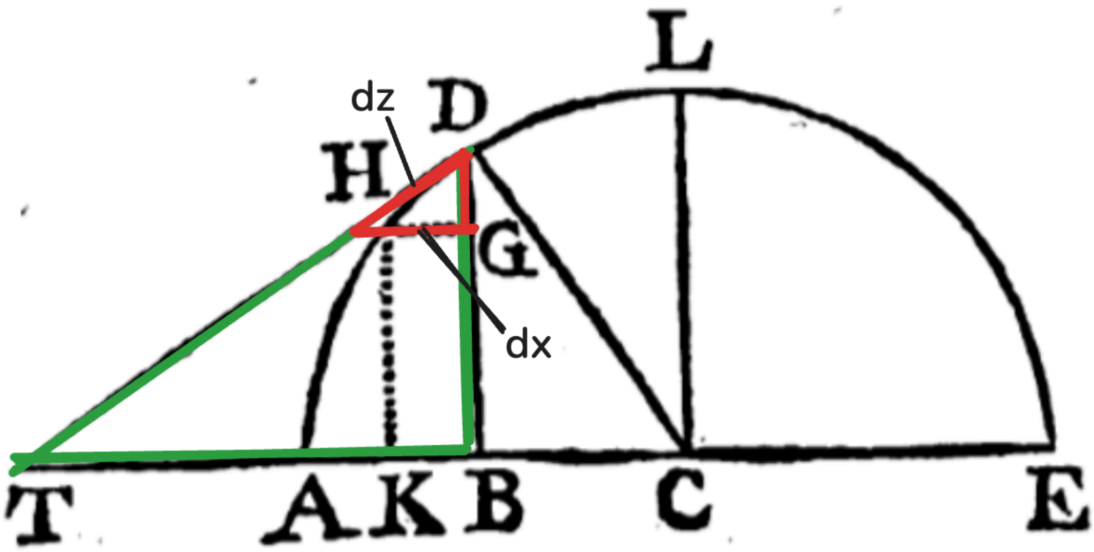


Figure 3: red and green triangles are similar

Next, we notice that the red and green triangles in Figure 4 are also similar, since both contain a right angle ($\angle TBD$ and $\angle DBC$) and $\angle CDB = \angle BTD$. The latter follows from the fact that in $\triangle DBT$, we can have $90° - \angle BDT = \angle BTD$, and since $\angle CDT = 90°$, we know that $\angle CDB = 90° - \angle BDT$, which we established was equal to $\angle BTD$.

Therefore:

$$\frac{DT}{BT} = \frac{DC}{BD} \tag{4}$$

Combining Equation 3 and Equation 4 to eliminate $\frac{DT}{BT}$ gives:

$$\frac{DH}{GH} = \frac{DC}{BD} = \frac{dz}{dx} \tag{5}$$

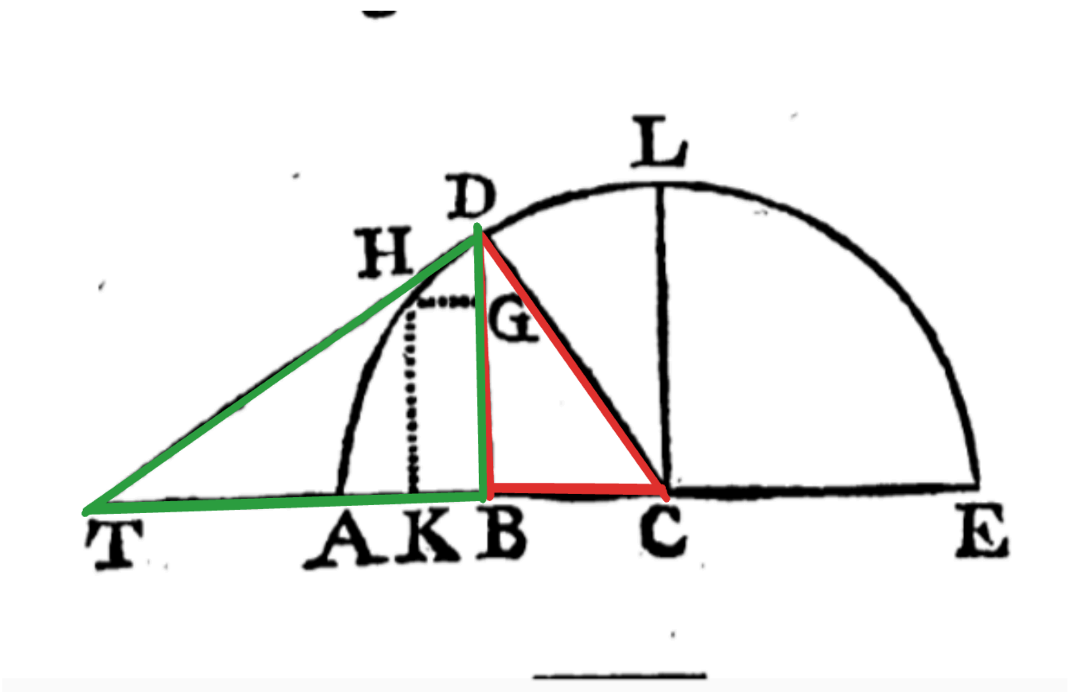


Figure 4: red and green triangles are similar

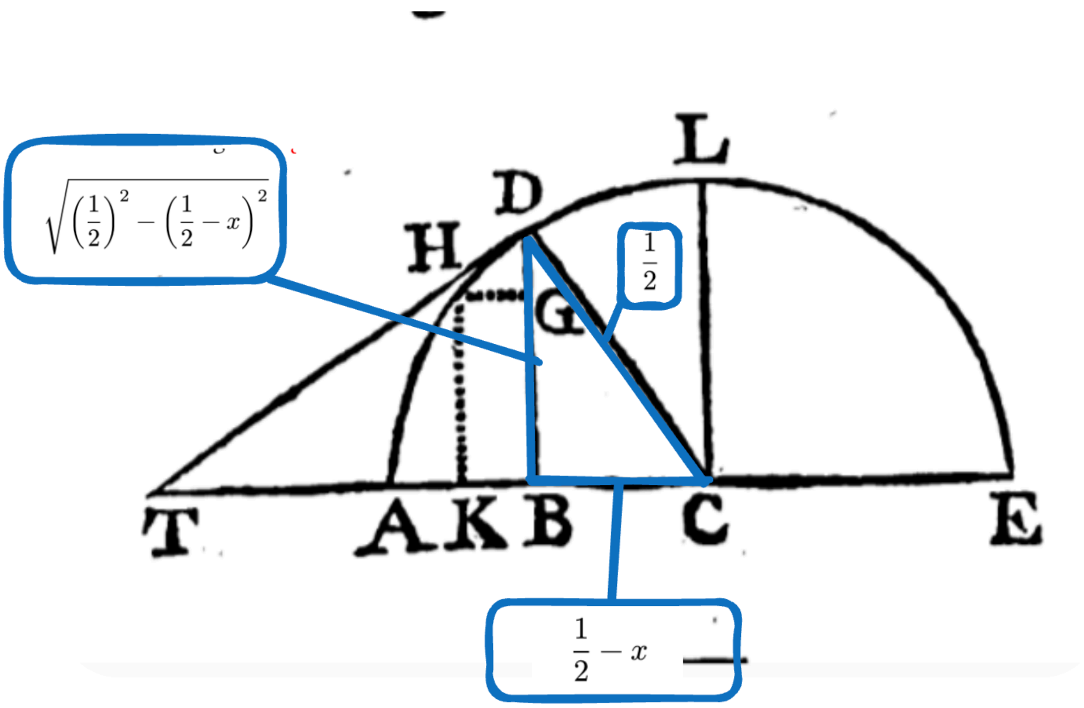


Figure 5: The Pythagorean theorem is used to find the value of the line BD

Next, by using the Pythagorean theorem on the blue triangle in Figure 5, and the fact that

$$DC = \frac{1}{2} \tag{6}$$

we find that:

$$BD = \sqrt{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2} - x\right)^2} = \sqrt{x - x^2} \tag{7}$$

Using Equation 6 and Equation 7 to substitute for DC and BD in Equation 5 gives:

$$\frac{dz}{dx} \equiv \frac{DH}{GH} = \frac{1}{2\sqrt{x - x^2}} \tag{8}$$

The arc length $z(x)$ can now be found by the inverse fluxional method (integration).

$$z(x) = \int \left(\frac{1}{2\sqrt{x - x^2}}\right) dx \tag{9}$$

In Newton's words, $\frac{1}{2\sqrt{x-x^2}}$ is the 'Moment' of the arc AD (or $z(x)$). Again, we are puzzled that he calls this the "Moment" rather than the "Fluxion" considering his own definitions for these terms. If this is really a Moment, where is the infinitesimal o?

We can perform the integral by first writing out the series expansion for $\frac{1}{2\sqrt{x-x^2}}$ using the binomial theorem, and then integrating term-by-term.

The binomial theorem is easier to apply if we first rewrite the Moment as:

$$\frac{1}{2\sqrt{x - x^2}} = \frac{1}{2}(x - x^2)^{-\frac{1}{2}} = \frac{x^{-\frac{1}{2}}}{2}(1 - x)^{-\frac{1}{2}} \tag{10}$$

Next we expand $(1 - x)^{-\frac{1}{2}}$ using Newton's recursive version of the Binomial theorem (Newton, 1676), namely:

$$(P + PQ)^{\frac{m}{n}} = \underbrace{P^{\frac{m}{n}}}_{\text{A}} + \underbrace{\frac{m}{n}AQ}_{\text{B}} + \underbrace{\frac{m-n}{2n}BQ}_{\text{C}} + \underbrace{\frac{m-2n}{3n}CQ}_{\text{D}} + ... \tag{11}$$

Where $A = P^{\frac{m}{n}}$, $P = 1$ and $Q = -x$, $m = -1$ and $n = 2$, we have:

$$\begin{aligned} A &= 1 \\ B &= \frac{1}{2}x \\ C &= -\frac{3}{4}\left(\frac{1}{2}x\right)x = -\frac{3}{8}x^2 \\ D &= \frac{-1-2(2)}{3 \times 2}\left(\frac{3}{8}x^2\right)x = \frac{5}{16}x^3 \end{aligned} \tag{12}$$

So:

$$(1 - x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{3}{8}x^2 + \frac{5}{16}x^3 + ... \tag{13}$$

Therefore:

$$\frac{1}{2\sqrt{x - x^2}} = \frac{x^{-\frac{1}{2}}}{2}(1 - x)^{-\frac{1}{2}} = \frac{x^{-\frac{1}{2}}}{2} + \frac{1}{4}x^{\frac{1}{2}} + \frac{3}{16}x^{\frac{3}{2}} + \frac{5}{32}x^{\frac{5}{2}} + ... \tag{14}$$

Now that we have obtained an expression for the moment of arc $\frac{dz}{dx}$ as a function of x (see Figure 6), we can integrate it term by term to get the arc length

$$\begin{aligned} \frac{dz}{dx} &= \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{4}x^{\frac{1}{2}} + \frac{3}{16}x^{\frac{3}{2}} + \frac{5}{32}x^{\frac{5}{2}} + ... \\ z(x) &= \int z(x)dx = x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{3}{2}} + \frac{3}{40}x^{\frac{5}{2}} + \frac{5}{112}x^{\frac{7}{2}} ... \end{aligned} \tag{15}$$

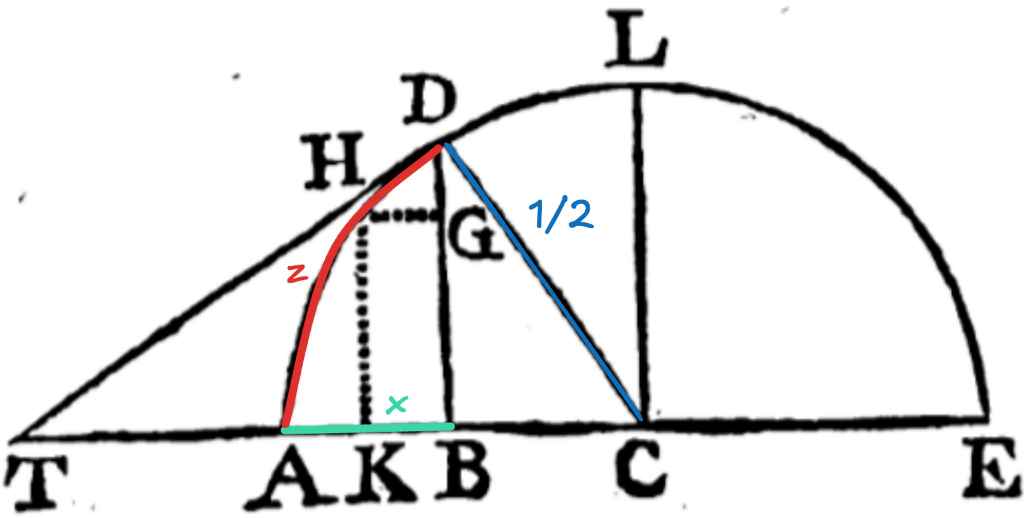


Figure 6: Finding $\frac{dz}{dx}$

39. Finding the Arc Sine as a power series

> 39. After the ſame Manner by ſuppoſing CB to be x, the Radius CA to be 1, you will find the Arch LD to be $x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c.

In §38, Newton showed that his method can find the arc length of a circle in terms of a coordinate x along its diameter. But he has not yet found the *arcsine*. For that, he needs to consider a unit circle, and change his independent variable x to measure a distance from the centre of the circle (Compare Figure 7 with Figure 6 to see the difference in geometry between §39 and §38):

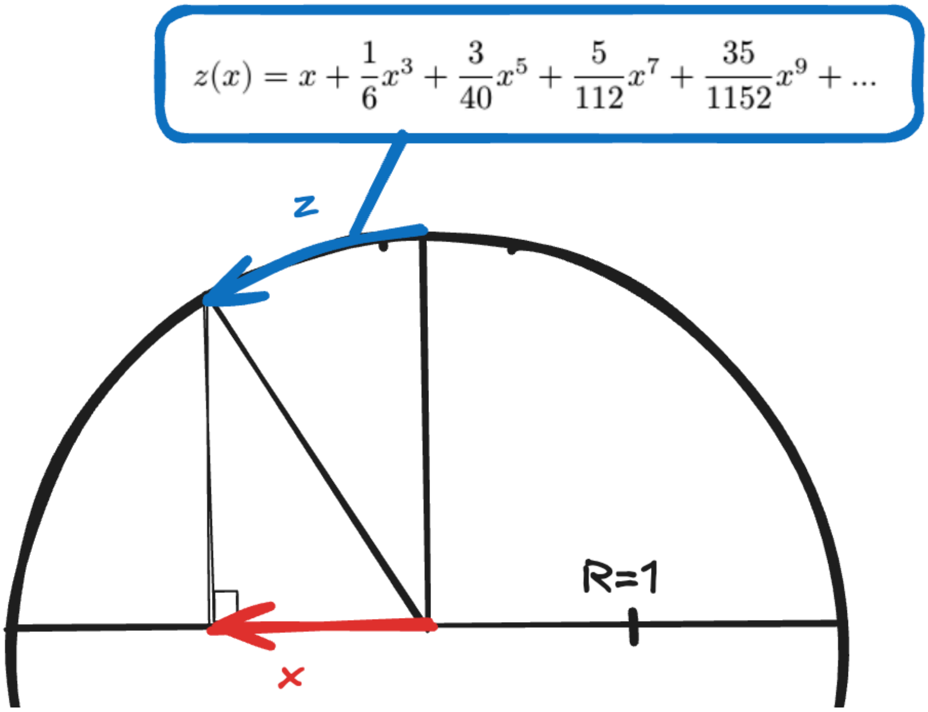


Figure 7: Summary of §39, (compare with Figure 2)

In contrast with §38, we now use a unit circle rather than a circle with diameter 1, so:

$$DC = 1 \tag{16}$$

The similar triangles argument that we used in §38 is still valid to establish the similarity between the infinitesimal triangle and DTB, as well as the similarity between DTB and DBC

But, instead of defining $AB = x$ we now have $BC = x$. By Pythagoras' theorem we obtain

$$BD = \sqrt{1 - x^2}. \tag{17}$$

This time we obtain a slightly different expression for the arc length (viz Equation 8):

$$\frac{dz}{dx} = \frac{1}{\sqrt{1 - x^2}} \tag{18}$$

which integrates to give:

$$z(x) = \int \left(\frac{1}{\sqrt{1 - x^2}} \right) dx. \tag{19}$$

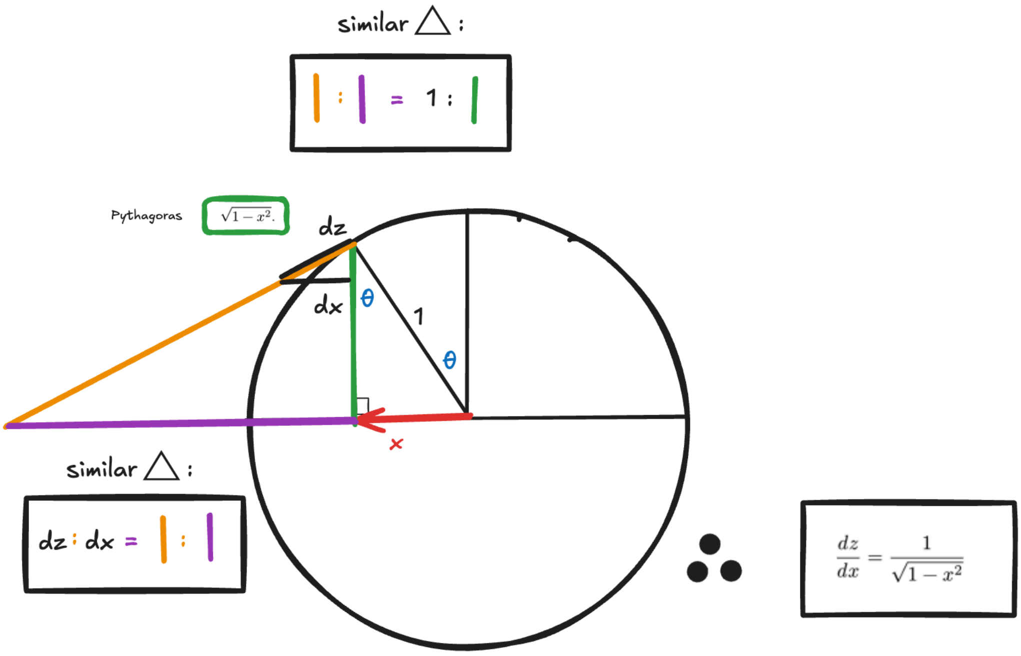


Figure 8: The similar triangles argument is given here without referring to any letters.

As before, applying the binomial theorem to the integrand allows us to re-write this expression as

$$z(x) = \int \left(1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8 + ...\right) dx \tag{20}$$

and integrating term by term gives

$$z(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + ... \tag{21}$$

Thus, we have obtained the power series expansion of the arcsine, through basic geometry, the binomial theorem, and integration of polynomials.

40. An aside on the Dimensionality of Unity

> 40. But it is to be remarked that that Unity which is put for the Moment, is a Superficies, when the Queſtion is about Solids; and a Line when about Superficies; and a Point when it is about Lines (as in this Example.) Neither am I afraid to ſpeak of Unity in Points, or Lines infinitely ſmall, ſince Geometers are wont now to conſider Proportions even in ſuch a Caſe, when they make uſe of the Methods of Indiviſibles.

The discussion on the dimensionality of the Moment (as being one less than the dimensionality of the generated quantity), casts into doubt whether it is correct to interpret Newton's Moments as infinitesimals. It is the *derivative* (or fluxion) not the infinitesimal line segment, that satisfies the property of being one dimension lower than the generated quantity.

41. An aside on Solids and Centers of Gravity

> 41. From theſe Things one may gueſs how one ought to proceed in inveſtigating the Superficies and Contents of Solids; and likewiſe the Centers of Gravity.

Regarding the contents, of solids, we remark that, if a two-dimensional x axis is needed to allow us to compare the area $\int y(x)dx$ with x in §37, would we seem to need a three-dimensional x-axis with area 1 to allow us to compare the volume of rotation of a curve against x. This seems fanciful, and we doubt whether Newton would take the requirement of ‘comparing like with like’ to that extreme.

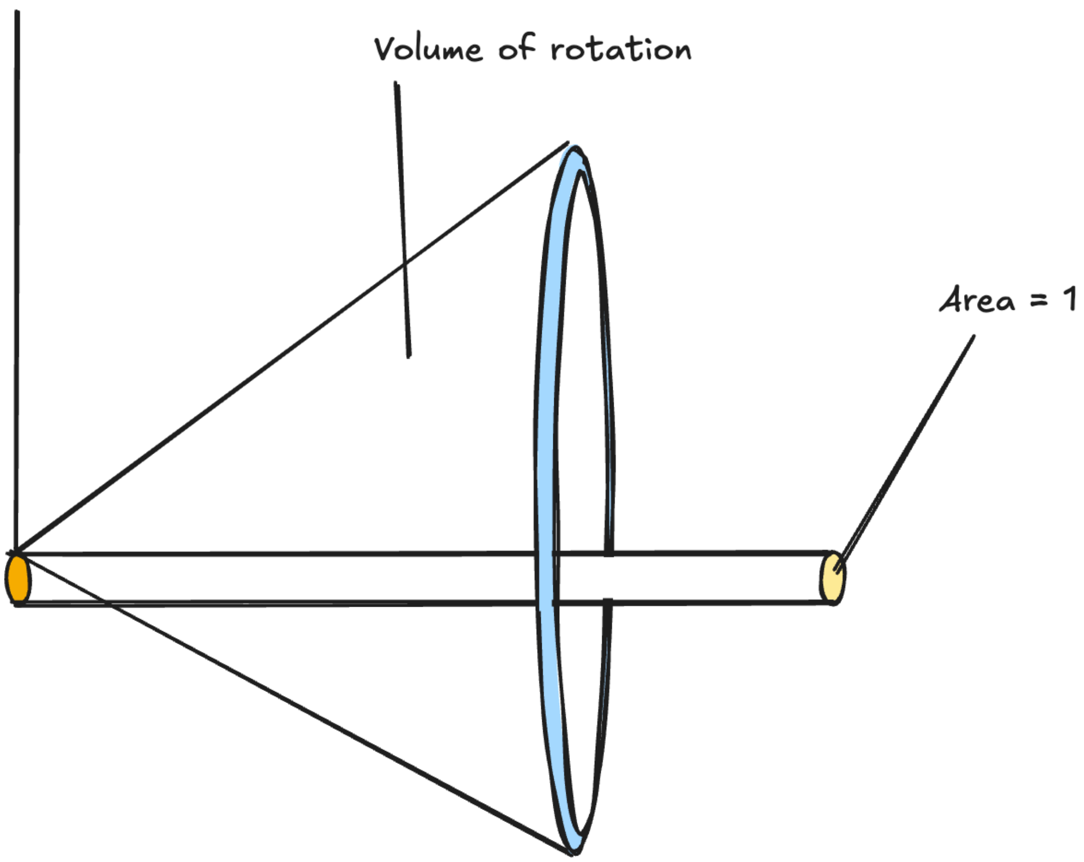


Figure 9: If Newton needs to construct a two-dimensional x-axis in order to rigorously compare how the area changes as a function of x, then would Newton have needed to construct a three-dimensional x-axis when considering a volume of rotation?

42. Inverting the power series for the Arc Sine by 'extracting the root'

> *To find the Converſe of theſe Things.*
>
> 42. But if upon the contrary, from the Area, or Length, &c. of any Curve being given, the Length of the Baſe AB be required, then you muſt extract the Root *x*, out of the Equations which have been found by the preceding Rules.

If we know the arc length z as a function of x (Equation 21), Newton says he can find an expression for x as a function of z. Since $x = \sin(z)$, this is equivalent to an expansion of $\sin(z)$ in terms of z.

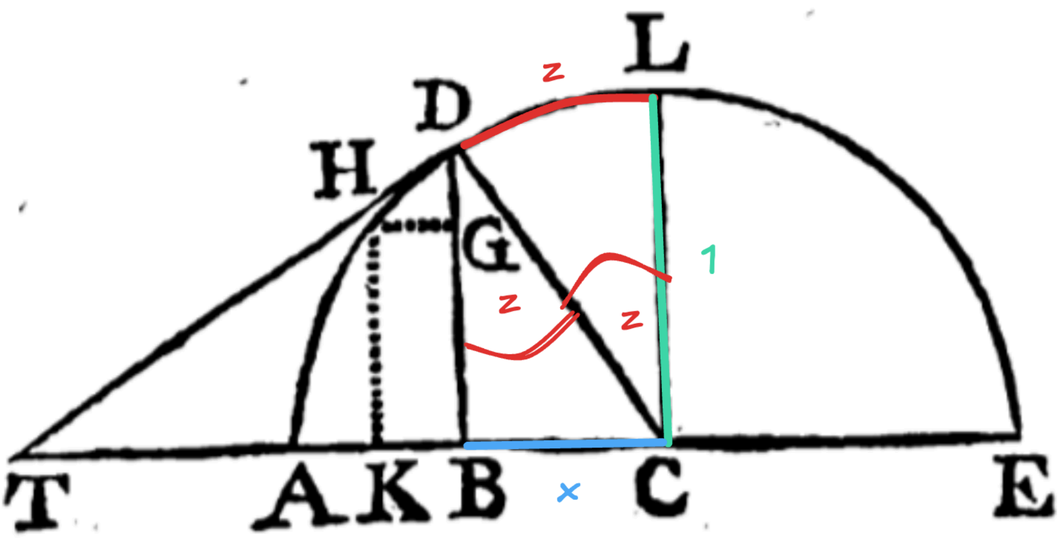


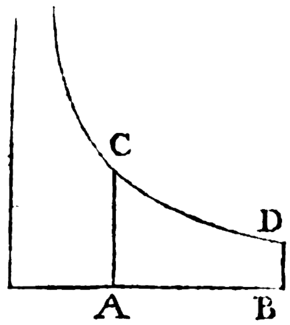
Figure 10: In a unit circle, the arc length z is also the angle z. Therefore, $x = \sin(z)$

In §43-44, the 'series inversion method' is introduced and a worked example is given. In §45, it will be applied to invert the arcsine.

43-44. Inverting $z = \ln(1 + x)$

Newton first demonstrates his series inversion method by inverting $z = \ln(1 + x)$ to find a power series expansion for what we today would call $x = e^z - 1$:

> 43. Thus if from the Area ABDC of the Hyperbola ($\frac{1}{1+x} = y$) given I wanted to inveſtigate the Baſe AB, calling the Area z, I extract the Root of this Equation z (ABCD) $= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$, &c. neglecting thoſe Terms in which x is of more Dimenſions than z is deſired in the Quotient.



It is no accident that he refers to x as a 'Quotient', since it appears to be the result of a process which closely resembles long-division, as we shall see.

So Newton starts with the power series:

$$\ln(1+x) \equiv z(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + ... \tag{22}$$

His aim is to find an expression like:

$$x(z) = \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 ... \tag{23}$$

Newton is aware that if he wants to find coefficients of the inverse power series, up to z^5 all he needs to do is invert infinite series of $z(x)$ truncated after the term x^5:

> As if I would have z to riſe to five Dimenſions only in the Quotient, I neglect all the Terms $-\frac{1}{6}x^6 + \frac{1}{7}x^7 - \frac{1}{8}x^8$, &c. and extract the Root of this only $\frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - z = 0$.

So, the infinite series:

$$z(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + ... \tag{24}$$

and the finite series:

$$z_5(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 \tag{25}$$

... will have a different inverse in general, but their inverses will be identical for the terms up to z^5. Therefore, Newton can focus on inverting just the finite series $z_5(x)$.

Newton next rewrites Equation 25 and puts all terms on the LHS:

$$-z + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 = 0 \tag{26}$$

He then rewrites Equation 26, stacking powers vertically:

$$0 = \begin{cases} +\frac{1}{5}x^5 \\ -\frac{1}{4}x^4 \\ +\frac{1}{3}x^3 \\ -\frac{1}{2}x^2 \\ + \ x \\ - \ z \end{cases} \tag{27}$$

It is apparent from Equation 27 (or Equation 26) that x has to equal z to lowest order, otherwise the equality cannot hold. Thus we express x as $x = z + p$, where p is taken to be $O(z^2)$.

Now we make our substitution

$$0 = \begin{cases} +\frac{1}{5}(z+p)^5 \\ -\frac{1}{4}(z+p)^4 \\ +\frac{1}{3}(z+p)^3 \\ -\frac{1}{2}(z+p)^2 \\ + \ z + p \\ - \ z \end{cases} \tag{28}$$

Then he expands Equation 28 selectively as follows:

$$0 = \begin{cases} +\frac{1}{5}z^5 + z^4p + \ldots & 5-5=0 \\ -\frac{1}{4}z^4 \ -z^3p + \frac{3}{2}z^2p^2 + \ldots. & 5-4=1 \\ +\frac{1}{3}z^3 \ +z^2p + zp^2 + \frac{1}{3}zp^3 + \ldots. & 5-3=2 \\ -\frac{1}{2}z^2 \ -zp - \frac{1}{2}p^2 + 0 & 5-2=3 \\ +z^1 + p & \\ -z^1 & \end{cases} \tag{29}$$

We have kept the first ignored terms in grey, and point out that they are all of a higher order in z than z^5. Newton uses the following rule to determine which terms to keep:

> "Rule; That after the firſt Term reſulting from each Quantity that is collateral to it, I add no more Terms upon the right Hand than the Index of the Dimenſion of that firſt Term wants Units of the Index of the greateſt Dimenſion. As in this Example, where the greateſt Dimenſion is 5, I neglect all the Terms after z^5, I put one after z^4, and two only after z^3. When the Root (x) to be extracted,

The *Units of the Index of the greatest dimension* is 5. The *Index of the Dimension of the First term* is 5 for the top row, and decreases by one for each row. Therefore $5 - 5 = 0$ and *no* additional terms are added after the z^5 term. The same logic leads us to conclude that one term is added after the z^4 term, and two after the z^3 term.

This means that terms like z^4p or z^2p^2, which are $\sim O(z^6)$, will be dropped. These terms are irrelevant because we are only seeking to find the coefficients up to z^5.

We are thus left with the following expression:

$$0 = \begin{cases} +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 - z^3p \\ +\frac{1}{3}z^3 + z^2p + zp^2 \\ -\frac{1}{2}z^2 - zp \quad - \frac{1}{2}p^2 \\ +z \quad\ + p \\ -z \end{cases} \tag{30}$$

Or, in Newton's original text:

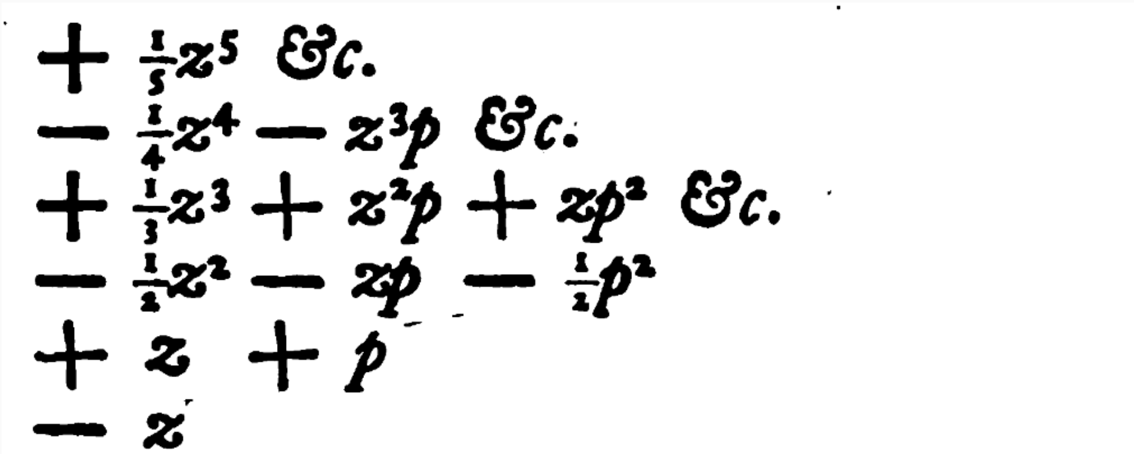

$+ \frac{1}{5}z^5$ &c.
$- \frac{1}{4}z^4 - z^3p$ &c.
$+ \frac{1}{3}z^3 + z^2p + zp^2$ &c.
$- \frac{1}{2}z^2 - zp - \frac{1}{2}p^2$
$+ z + p$
$- z$

Figure 11: (Newton & Stewart, 1745, p. 337)

Since we are looking at the lowest term in the expansion for p, we find that Equation 30 can only hold if:

$$p = \frac{1}{2}z^2 + O(z^3) \tag{31}$$

So we have now found our second order expansion of $x(z)$:

$$x = z + \frac{z^2}{2} + O(z^3) \tag{32}$$

Newton's table up until this point looks like this:

$z + p = x$	$+\frac{1}{5}x^5$ $-\frac{1}{4}x^4$ $+\frac{1}{3}x^3$ $-\frac{1}{2}x^2$ $+x$ $-z$	$+\frac{1}{5}z^5$ &c. $-\frac{1}{4}z^4 - z^3p$ &c. $+\frac{1}{3}z^3 + z^2p + zp^2$ &c. $-\frac{1}{2}z^2 - zp - \frac{1}{2}p^2$ $+z + p$ $-z$
$\frac{1}{2}z^2 + q = p$		

Figure 12: The steps to find the first and second terms in the inversion of $\ln(1+x)$

Now Newton returns to Equation 30, and rewrites it in order of decreasing dimensions of p and then z:

$$0 = \begin{cases} zp^2 \\ -\frac{1}{2}p^2 \\ -z^3p \\ +z^2p \\ -zp \\ +p \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases} \tag{33}$$

In Newton's original text, this looks like this:

> | | | |
> |---|---|---|
> | $z + p = x$ | $+\frac{1}{5}x^5$
$-\frac{1}{4}x^4$
$+\frac{1}{3}x^3$
$-\frac{1}{2}x^2$
$+x$
$-z$ | $+\frac{1}{5}z^5$ &c.
$-\frac{1}{4}z^4 - z^3p$ &c.
$+\frac{1}{3}z^3 + z^2p + zp^2$ &c.
$-\frac{1}{2}z^2 - zp - \frac{1}{2}p^2$
$+z + p$
$-z$ |
> | $\frac{1}{2}z^2 + q = p$ | $+zp^2$
$-\frac{1}{2}p^2$
$-z^3p$
$+z^2p$
$-zp$
$+p$
$+\frac{1}{5}z^5$
$-\frac{1}{4}z^4$
$+\frac{1}{3}z^3$
$-\frac{1}{2}z^2$ | |

Now, he substitutes $p = \frac{1}{2}z^2 + q$, where q is assumed to be of the form $q = O(z^3) + ...$ He uses the same rules to determine which terms to keep

$$0 = \begin{cases} +z\left(\frac{1}{2}z^2 + q\right)^2 \\ -\frac{1}{2}\left(\frac{1}{2}z^2 + q\right)^2 \\ -z^3\left(\frac{1}{2}z^2 + q\right) \\ +z^2\left(\frac{1}{2}z^2 + q\right) \\ -z\left(\frac{1}{2}z^2 + q\right) \\ +\left(\frac{1}{2}z^2 + q\right) \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases} = \begin{cases} \frac{1}{4}z^5 + z^3q & 5 - 5 = 0 \\ -\frac{1}{8}z^4 \; -\frac{1}{2}z^2q & 5 - 4 = 1 \\ -\frac{1}{2}z^5 - z^3q & 5 - 5 = 0 \\ +\frac{1}{2}z^4 + z^2q & 5 - 4 = 1 \\ -\frac{1}{2}z^3 - qz & 5 - 3 = 2 \\ +\frac{1}{2}z^2 + q & \\ +\frac{1}{5}z^5 & \\ -\frac{1}{4}z^4 & \\ +\frac{1}{3}z^3 & \\ -\frac{1}{2}z^2 & \end{cases} \tag{34}$$

Since q only enters linearly in Equation 34, we have a simple closed-form solution for q in terms of z:

$$x = z + \frac{1}{2}z^2 + q \tag{35}$$

where q is found by solving for q in Equation 34:

$$q = \frac{\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5}{1 - z + \frac{1}{2}z^2} \tag{36}$$

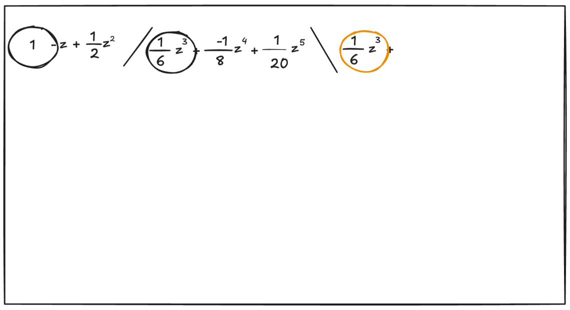
Thus we have worked out a closed-form solution for the inverse of the finite series (Equation 25). q can be found by polynomial long division, as Newton explains in §44.2:

> 2. When I ſee that *p*, *q*, or *r*, &c. in the laſt reſulting Equation, is found of one Dimenſion only, I ſeek it's Value, that is to ſay the remaining Terms, which are ſtill to be added to the Quotient, by means of Diviſion; as you ſee done here.

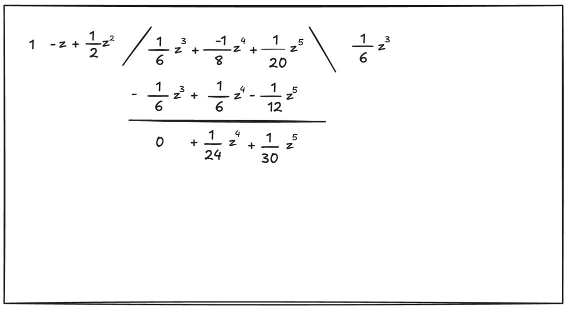
Here Newton refers us to the bottom of his table (Figure 13) which contains the result of the long division:

> $1 - z + \frac{1}{2}z^2)\ \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5\ (\frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$

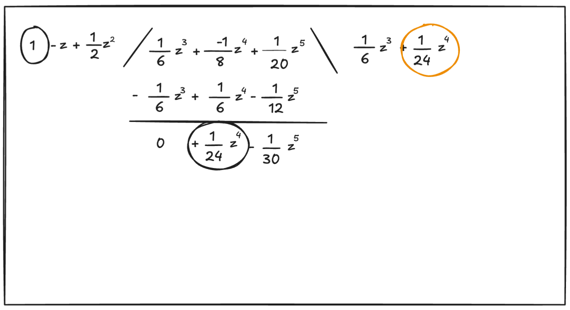
We have recreated the long division below:



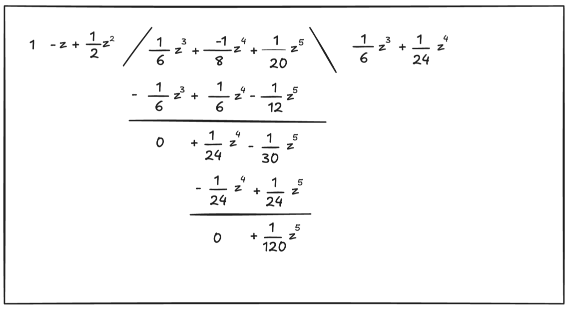
Step 1: Dividing $\frac{1}{6}z^3$ by 1



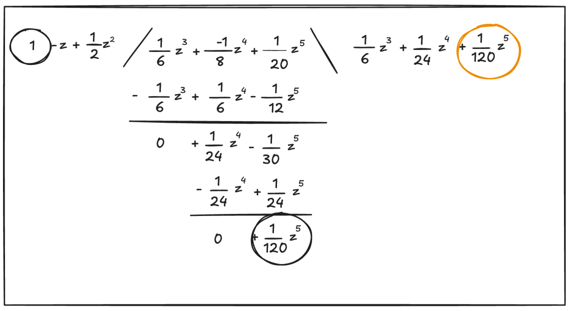
Step 2: Calculating the remainder



Step 3: Dividing $\frac{1}{24}z^4$ by 1



Step 4: Calculating the remainder



Step 5: Dividing $\frac{1}{120}z^5$ by 1 to yield the final term

At this point we can stop, remembering that we sought the coefficients up to z^5. We could, of course, continue the division beyond z^5 but these terms would be meaningless as the inverse of the infinite series $z(x)$ and the inverse of the finite series $z_5(x)$ cease to be identical after the z^5 term.

We have found q up to z^5:

$$q = \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 + O(z^7) \tag{37}$$

We can plug this into our existing expansion for x:

$$x = z + \frac{1}{2}z^2 + q \tag{38}$$

Thus, the first 5 terms of the expansion of $e^z - 1$ is:

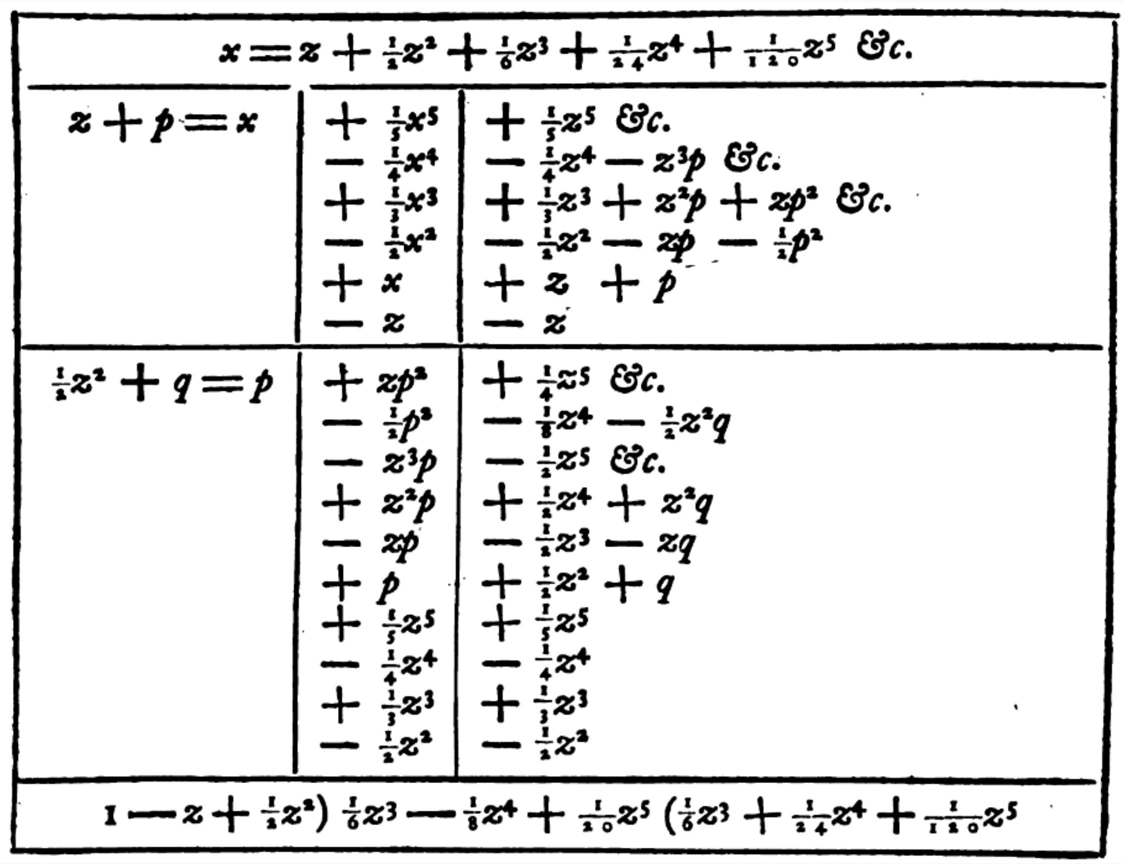
$$x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 + ... \equiv e^z - 1 \tag{39}$$

If we wanted to find the next coefficient, we would have had to start with one more power of x in the series for z, namely:

$$z_6(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 \tag{40}$$

And use the same method.

Newton's completed table is shown below:



$x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$ &c.

$z + p = x$	$+\frac{1}{5}x^5$ $-\frac{1}{4}x^4$ $+\frac{1}{3}x^3$ $-\frac{1}{2}x^2$ $+x$ $-z$	$+\frac{1}{5}z^5$ &c. $-\frac{1}{4}z^4 - z^3p$ &c. $+\frac{1}{3}z^3 + z^2p + zp^2$ &c. $-\frac{1}{2}z^2 - zp - \frac{1}{2}p^2$ $+z + p$ $-z$
$\frac{1}{2}z^2 + q = p$	$+zp^2$ $-\frac{1}{2}p^2$ $-z^3p$ $+z^2p$ $-zp$ $+p$ $+\frac{1}{5}z^5$ $-\frac{1}{4}z^4$ $+\frac{1}{3}z^3$ $-\frac{1}{2}z^2$	$+\frac{1}{4}z^5$ &c. $-\frac{1}{8}z^4 - \frac{1}{2}z^2q$ $-\frac{1}{2}z^5$ &c. $+\frac{1}{2}z^4 + z^2q$ $-\frac{1}{2}z^3 - zq$ $+\frac{1}{2}z^2 + q$ $+\frac{1}{5}z^5$ $-\frac{1}{4}z^4$ $+\frac{1}{3}z^3$ $-\frac{1}{2}z^2$

$1 - z + \frac{1}{2}z^2)\ \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5\ (\frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$

Figure 13: Series inversion algorithm for $z = \ln(1 + x)$. The answer is at the top. The second and third rows show the results of plugging in the first-order and second-order result and working out the remainder. The last row shows us how the z^3, z^4, and z^5 terms are calculated by long division.

This method is closely related to the modern method of undetermined coefficients (see Appendix), except the latter makes it more explicit that we are assuming an equation of the form $\sum_i^\infty C_i x^i$, whereas Newton's method leaves that implicit. We believe that the parallel with long division with decimal numbers would have made the explicit assumption of an equation of a certain form to be pedantic. After all, when dividing two decimal numbers, we do not start by *assuming* an answer of a certain form (with a decimal point and infinite digits after the decimal). We simply work it out. Similarly, Newton just works out the series without worrying about stating its form to begin with.

45. Inverting $z = \sin^{-1}(x)$ to find the sine.

Newton states that we can apply his series inversion method (worked example in §43-44) to his expression of the arcsine (Equation 41) to get a power series expression of the sine:

> "45. If from the Arch αD given the Sine AB was required; I extract the Root of the Equation found above, *viz.* $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$ (it being ſuppoſed that $AB = x$, $\alpha D = z$, and $A\alpha = 1$) by which I find $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$ &c.

It may seem a little strange that Newton does not feel the need to explicitly give the steps needed to invert the arcsine expansion (Equation 41):

$$\sin^{-1}(x) \equiv z(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + ... \tag{41}$$

We will apply the method that he details in §43-44 to check that you indeed get the answer Newton states in §45. Equation 41 differs from the expansion of Equation 22 in that it contains only odd powers, and only positive coefficients. The coefficients themselves are more complicated, however. This time we want a solution up to the power of z^9, so we must consider the series truncated up to the x^9 term:

$$z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 \tag{42}$$

As before, the first order inverse is $x = z$, as Equation 42 would not hold if $x \neq z$ to lowest order. We find the next order by substituting $x = z + p$ into Equation 42. We rearrange it first into decreasing powers of x

$$0 = \begin{cases} +\frac{35}{1152}x^9 \\ +\frac{5}{112}x^7 \\ +\frac{3}{40}x^5 \\ +\frac{1}{6}x^3 \\ +x \\ -z \end{cases} \tag{43}$$

Now we substitute the expansion $x = z + p$

$$0 = \begin{cases} +\frac{35}{1152}(z+p)^9 \\ +\frac{5}{112}(z+p)^7 \\ +\frac{3}{40}(z+p)^5 \\ +\frac{1}{6}(z+p)^3 \\ +x \\ -z \end{cases} \tag{44}$$

Again, we don't need to expand all the terms, only the ones that will be of a lower order than x^9, which we have done below.

$$
0 = \begin{cases}
+\frac{35}{1152}(z^9 + 9z^8p) + \dots & \frac{9-9}{2} = 0 \\
+\frac{5}{112}\left(z^7 + \boxed{7z^6p} + 28z^5p^2\right) + \dots & \frac{9-7}{2} = \boxed{1} \\
+\frac{3}{40}\left(z^5 + \boxed{5z^4p + 10z^3p^2} + 10z^2p^3\right) + \dots & \frac{9-5}{2} = \boxed{2} \\
+\frac{1}{6}\left(z^3 + \boxed{3z^2p + 3zp^2 + p^3}\right) + \dots & \frac{9-3}{2} = \boxed{3} \\
+z + p \\
-z
\end{cases} \tag{45}
$$

Again, we have added the next (ignored) term in grey.

Newton explains how we must modify his rule about which terms to keep for the case of an even or odd power series such as this one:

> after z^4, and two only after z^3. When the Root *(x)* to be extracted, is every where of even or odd Dimenſions, let this be the Rule: That after the firſt Term, reſulting from each Quantity which is collateral to it, you add no more Terms towards the Right Hand, than what the Index of the Dimenſion of that firſt Term, wants Pairs of Units of the Index of the higheſt Dimenſion; or no more than what it wants Ternaries of Units, when the Indexes of the Dimenſions of x differ by three Units; and ſo in others.

The *Index of the Highest Dimension* is 9. The *Index of the dimension of the first term* is 9 for the first row and drops by 2 in each row. “wants Pairs of Units” means we have to divide the difference by two to get the number of additional units on the right hand side.

To summarize, ignoring orders higher than $O(z^9)$ allows us to consider the (much simplified) Equation 46 rather than the full Equation 44

$$
0 = \begin{cases}
+\frac{35}{1152}z^9 + \dots \\
+\frac{5}{112}(z^7 + 7z^6p) + \dots \\
+\frac{3}{40}(z^5 + 5z^4p + 10z^3p^2) + \dots \\
+\frac{1}{6}(z^3 + 3z^2p + 3zp^2 + p^3) \\
+z + p \\
-z
\end{cases} \tag{46}
$$

It is clear that Equation 46 can only hold if:

$$
p = -\frac{1}{6}z^3 + O(z^5) \tag{47}
$$

For something must cancel the $\frac{1}{6}z^3$ in Equation 46. We have thus found the third-order approximation:

$$
x = z - \frac{1}{6}z^3 \tag{48}
$$

We now rewrite Equation 45 in terms of decreasing powers of p and z, just like we did before (Equation 33) when calculating the inverse of $\ln(1+x)$ in §43.

$$0 = \begin{cases} +\frac{1}{6}p^3 \\ +\frac{3}{4}z^3p^2 \\ +\frac{1}{2}zp^2 \\ +\frac{35}{112}z^6p \\ +\frac{15}{40}z^4p \\ +\frac{1}{2}z^2p \\ p \\ +\frac{35}{1152}z^9 \\ +\frac{5}{112}z^7 \\ +\frac{3}{40}z^5 \\ +\frac{1}{6}z^3 \\ \cancel{+z} \\ \cancel{-z} \end{cases} \tag{49}$$

We now use the substitution $p = -\frac{1}{6}z^3 + q$ into Equation 49 where we assume that the leading coefficient of q is $O(z^5)$. Again, we use Newton's rule to discard terms of higher order than z^9:

$$0 = \left\{ \begin{array}{ll} +\frac{1}{6}\left(-\frac{1}{6^3}z^9\right) + \ldots & \frac{9-9}{2} = 0 \\ +\frac{3}{4}z^3\left(\frac{1}{36}z^6\right) + \ldots & \frac{9-9}{2} = 0 \\ +\frac{1}{2}z\left(\frac{1}{36}z^6 \boxed{-\frac{1}{3}qz^3} + \ldots\right) & \frac{9-7}{2} = \boxed{1} \\ +\frac{35}{112}z^6\left(-\frac{1}{6}z^3\right) + \ldots & \frac{9-9}{2} = 0 \\ +\frac{15}{40}z^4\left(-\frac{1}{6}z^3 + \boxed{q}\right) & \frac{9-7}{2} = \boxed{1} \\ +\frac{1}{2}z^2\left(-\frac{1}{6}z^3 + q\right) & \frac{9-5}{2} = 2 \\ -\frac{1}{6}z^3 + q & \\ +\frac{35}{1152}z^9 & \\ +\frac{5}{112}z^7 & \\ +\frac{3}{40}z^5 & \\ +\frac{1}{6}z^3 & \end{array} \right. \tag{50}$$

Expanding the brackets:

$$
0=\begin{cases}
-\frac{1}{1296}z^9+... \\
+\frac{1}{48}z^9+... \\
+\frac{1}{72}z^7 & -\frac{1}{6}qz^4+... \\
-\frac{35}{672}z^9+... \\
-\frac{15}{240}z^7 & +\frac{15}{40}qz^4+... \\
-\frac{1}{12}z^5 & +\frac{1}{2}z^2q \\
-\frac{1}{6}z^3+q \\
+\frac{35}{1152}z^9 \\
+\frac{5}{112}z^7 \\
+\frac{3}{40}z^5 \\
+\frac{1}{6}z^3
\end{cases}
\tag{51}
$$

Adding like terms:

$$
0=\begin{cases}
\left(-\frac{1}{1296}+\frac{1}{48}-\frac{35}{672}+\frac{35}{1152}\right)z^9 \\
\left(+\frac{1}{72}-\frac{15}{240}+\frac{5}{112}\right)z^7 \\
\left(+\frac{15}{40}-\frac{1}{6}\right)qz^4 \\
\left(-\frac{1}{12}+\frac{3}{40}\right)z^5 \\
+\frac{1}{2}z^2q \\
\cancel{-\frac{1}{6}z^3}+q \\
\cancel{+\frac{1}{6}z^3}
\end{cases}
\tag{52}
$$

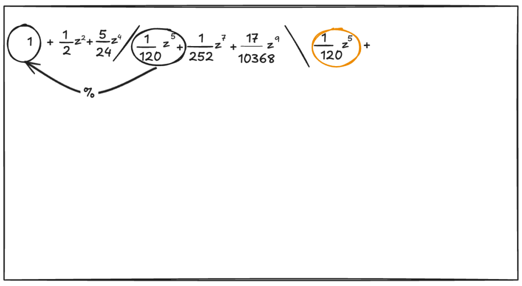
At this point we find that our equation is linear in q, which means we can solve it using polynomial division:

$$
0=\begin{cases}
-\frac{17}{10368}z^9 \\
-\frac{1}{252}z^7 \\
+\frac{5}{24}qz^4 \\
-\frac{1}{120}z^5 \\
+\frac{1}{2}z^2q \\
q
\end{cases}
\tag{53}
$$

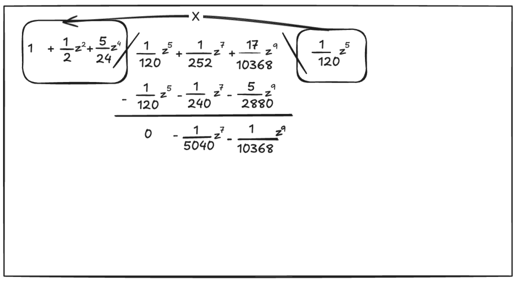
$$
\frac{1}{120}z^5+\frac{1}{252}z^7+\frac{17}{10368}z^9=q\left(1+\frac{1}{2}z^2+\frac{5}{24}z^4\right)
\tag{54}
$$

$$
\frac{\frac{1}{120}z^5+\frac{1}{252}z^7+\frac{17}{10368}z^9}{1+\frac{1}{2}z^2+\frac{5}{24}z^4}=q
\tag{55}
$$

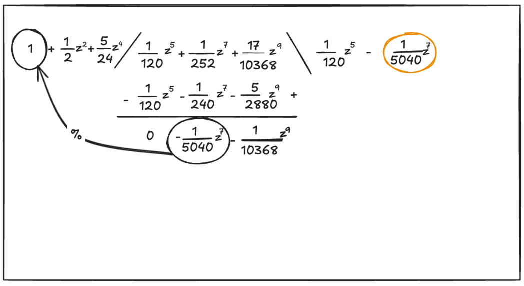
We can calculate the remaining terms by straightforward polynomial division:



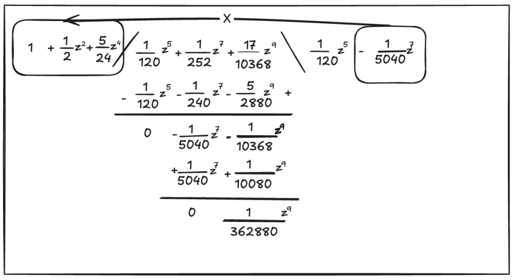
Step 1: Dividing $\frac{1}{120}z^5$ by 1



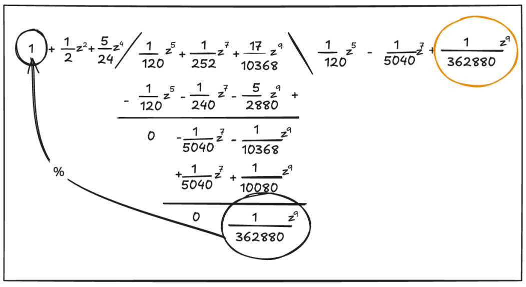
Step 2: Calculating the remainder



Step 3: Dividing $-\frac{1}{5040}z^7$ by 1



Step 4: Calculating the remainder



Step 5: Dividing $\frac{1}{362880}z^9$ by 1

By long division:

$$q = \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 + O(z^{11}) \tag{56}$$

And therefore

$$\begin{aligned} x &= z - \frac{1}{6}z^3 + q \\ &= z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 + O(z^{11}) \end{aligned} \tag{57}$$

We recognize this as the first few terms of the series expansion of the sine:

$$x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \tag{58}$$

We must point out that whereas we can continue the long division as long as we like, the resulting terms will have *nothing* to do with the sine. Instead they will be the continuation of the infinite series expansion of the inverse of the *truncated* expansion of the arcsine (Equation 42). To calculate the z^{13} term we would have had to start with the expansion of the arcsine $z(x)$ up to order x^{13} and perform the same procedure.

Newton does not give us a table for his inversion of $\sin^{-1}(x)$ as he did for $\ln(1+x)$. But if he had, we can be fairly confident it would have looked something like this (viz. Figure 13):

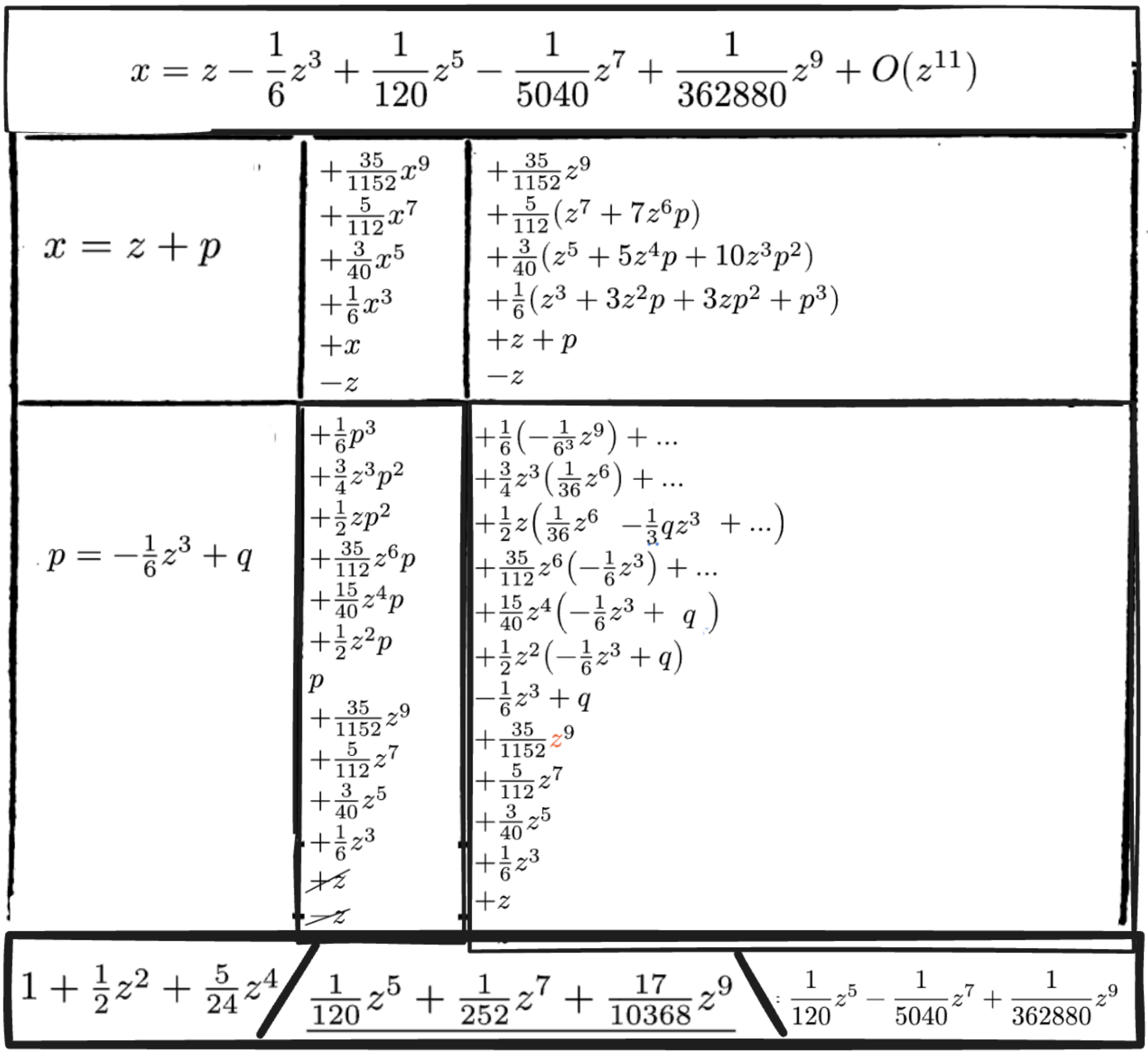
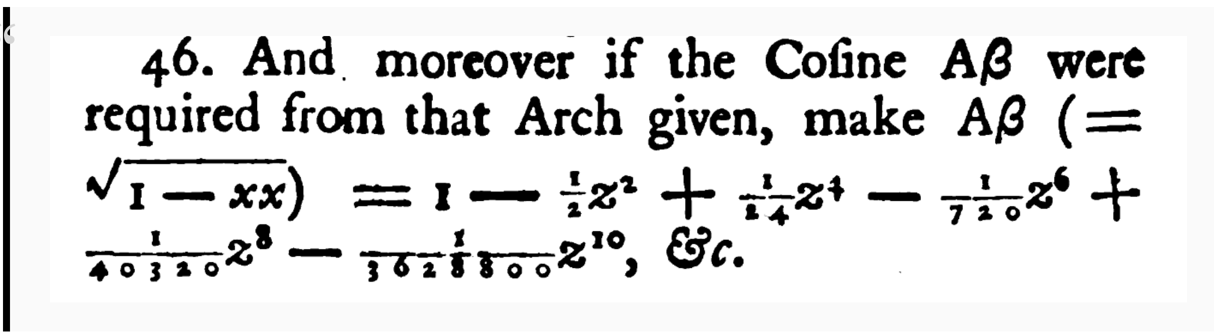


Figure 14: Reconstructed series inversion table for the arcsine. Compare with Figure 13.

46. Finding the power series expansion of the cosine


"46. And moreover if the Cofine Aβ were required from that Arch given, make Aβ ($=\sqrt{1-xx}$) $= 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6 + \frac{1}{40320}z^8 - \frac{1}{3628800}z^{10}$, &c.

Finally, from the power series for the sine, Newton derives the power series for the cosine. He does this making use of the identity

$$\cos(z) = \sqrt{1 - \sin^2(z)}, \tag{59}$$

which follows from the fact that in Figure 15, $BD = \cos(z)$. We can then apply Pythagoras' theorem as follows:

$$BD = \sqrt{CD^2 - BC^2} = \sqrt{1 - x^2} \tag{60}$$

Substituting $x = \sin(z)$ gives:

$$BD = \sqrt{1 - \sin^2(z)} \tag{61}$$

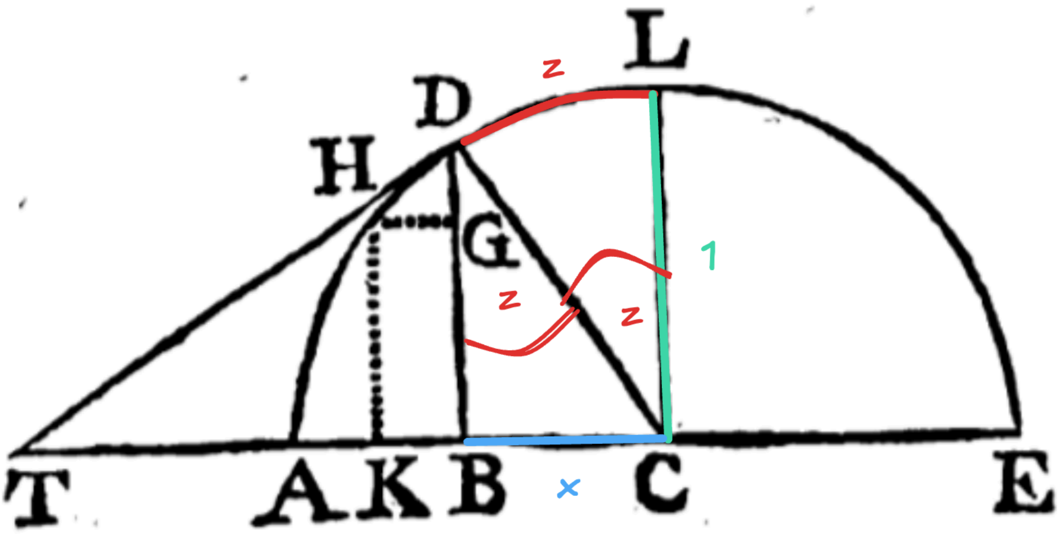


Figure 15: Repetition of Figure 11. From this figure it should be apparent that the length $BD = \cos(z)$.

In order to find a power series for the cosine, we can therefore take the power series for $\sqrt{1 - x^2}$, and substitute for each x in this series the power series for $\sin(z)$. Therefore, our next step is to express $\sqrt{1 - x^2}$ as a power series, which we do with the binomial theorem:

$$(1 - x^2)^{\frac{1}{2}} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 + \ldots \tag{62}$$

Considering only the first three terms will suffice for our present purposes. The next step will be to plug our power series for the sine into each x in Equation 62. This gives:

$$\begin{aligned} \cos(z) = 1 - \frac{1}{2}\left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \ldots\right)^2 - \frac{1}{8}\left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \ldots\right)^4 \\ - \frac{1}{16}\left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \ldots\right)^6 + \ldots \end{aligned} \tag{63}$$

Now all we need to do is order the z terms. The first and second terms, in z^0 and z^2, are easily found since they both appear only once. Therefore we have:

$$\cos(z) = 1 - \frac{1}{2}z^2 + \ldots \tag{64}$$

The next term will be on the order of z^4, so we need to collect all z^4 terms in the expansion. This gives us

$$\cos(z) = 1 - \frac{1}{2}z^2 - \frac{1}{2}\cdot -\frac{1}{6}z^3 \cdot z - \frac{1}{2}\cdot z \cdot -\frac{1}{6}z^3 - \frac{1}{8}z^4 + ... \\ = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 + ... \tag{65}$$

Next, we collect all the z^6 terms, which gives us this:

$$\cos(z) = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - 2\cdot\frac{1}{2}z\cdot\frac{1}{120}z^5 - \frac{1}{2}\cdot -\frac{1}{6}z^3 \cdot -\frac{1}{6}z^3 - 4\cdot\frac{1}{8}z\cdot z\cdot z\cdot -\frac{1}{6}z^3 - \frac{1}{16}z^6 \\ = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6 + ... \tag{66}$$

Newton continues this procedure for two more terms, thus finding the coefficients for the z^8 and z^{10} terms as well.

47. Recognising the factorial pattern

The following section makes it clear that Newton saw the factorial pattern of the sine and cosine series, which today we would write as:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \tag{67}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \tag{68}$$

Newton did not use these symbols as the sigma summation notation ($\sum$) was introduced by Euler in 1755 (Campbell, n.d.), and the factorial was introduced in 1808 by Christian Kramp (O'Connor & Robertson, 1997).

> *Concerning the Continuation of the Series of the Progreſſions.*
>
> 47. Let it be obſerved here, by the bye, that when 5 or 6 Terms of thoſe Roots are known, they may be continued at Pleaſure for moſt Part, by obſerving the Analogy of the Progreſſion.
>
> Thus you may continue this $x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$, &c. by dividing the laſt Term by the following Numbers in Order, 2, 3, 4, 5, 6, 7, &c.
>
> And this $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7$, &c. by dividing by theſe Numbers 2×3, 4×5, 6×7, 8×9, 10×11, &c.
>
> And

> And this $x = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6$, &c. by theſe $1 \times 2, 3 \times 4, 5 \times 6, 7 \times 8, 9 \times 10$, &c.
> And this $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c. by multiplying by theſe, *viz.* $\frac{1\times1}{2\times3}, \frac{3\times3}{4\times5}, \frac{5\times5}{6\times7}, \frac{7\times7}{8\times9}$, &c. And ſo in others.

Concluding Remarks

We have seen that Newton used elementary geometry, the binomial theorem, and integration to derive the series for the sine. Notably, Newton arrives at these results without Taylor series. His recognition of the factorial pattern in §47 suggests he understood the deeper structure of the result, even without the notation to express it cleanly.

Two open questions remain. First, why does Newton uses "Moment" when "Fluxion" would do just as well (§37 and §38)? Can Moments and Fluxions be identical for Newton? Second, how would Newton's x-axis adapt to a Volume of rotation, given that it becomes two-dimensional when treating an area under a curve.

Appendix: The 'method of undetermined coefficients'

Newton's method is computationally equivalent to the more modern "method of undetermined coefficients", which is detailed below. We start with the $z(x)$ expression we had before:

$$z(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5.. \tag{69}$$

This time we explicitly assume that we can write x as a series expansion in terms of z:

$$x(z) = \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 \tag{70}$$

And we search for the coefficients term-by-term by repeatedly substituting Equation 70 into Equation 69. Our single equation turns into 5 equations in 5 variables, one equation for each power of z:

$$\begin{aligned} z(x) = &\left(\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5\right) \\ &-\frac{1}{2}\left(\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5\right)^2 \\ &+\frac{1}{3}\left(\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5\right)^3 \\ &-\frac{1}{4}\left(\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5\right)^4 \\ &+\frac{1}{5}\left(\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5\right)^5 \\ &.. \end{aligned} \tag{71}$$

The powers of z^1 on LHS and RHS must be equal:

$$\alpha = 1 \tag{72}$$

The powers of z^2 on LHS and RHS must equal:

$$
\begin{aligned}
0 &= \beta + -\frac{1}{2}z^2 \\
\beta &= \frac{1}{2}
\end{aligned}
\tag{73}
$$

The same goes for the powers of z^3:

$$
\begin{aligned}
0 &= \gamma + -\frac{1}{2}(2\beta\alpha) + \left(\frac{1}{3}\right)\alpha^3 \\
\gamma &= \frac{1}{2} - \frac{1}{3} \\
\gamma &= \frac{1}{6}
\end{aligned}
\tag{74}
$$

The same goes for the powers of z^4:

$$
\begin{aligned}
0 &= \delta + \left(-\frac{1}{2}\right)(\beta^2 + 2\alpha\gamma) + \left(\frac{1}{3}\right)(3\alpha^2\beta) - \left(\frac{1}{4}\right)\alpha^4 \\
0 &= \delta + \left(-\frac{1}{2}\right)\left(\left(\frac{1}{2}\right)^2 + 2(1)\left(\frac{1}{6}\right)\right) + \left(\frac{1}{3}\right)\left(3(1)^2\left(\frac{1}{2}\right)\right) - \left(\frac{1}{4}\right)(1)^4 \\
0 &= \delta + \left(-\frac{1}{2}\right)\left(\frac{1}{4} + \frac{1}{3}\right) + \frac{1}{2} - \frac{1}{4} \\
0 &= \delta + \left(-\frac{1}{2}\right)\left(\frac{3}{12} + \frac{4}{12}\right) + \frac{12}{24} - \frac{6}{24} \\
0 &= \delta + \left(-\frac{1}{2}\right)\left(\frac{7}{12}\right) + \frac{12}{24} - \frac{6}{24} \\
0 &= \delta - \frac{7}{24} + \frac{12}{24} - \frac{6}{24} \\
0 &= \delta - \frac{1}{24} \\
\delta &= \frac{1}{24}
\end{aligned}
\tag{75}
$$

And finally, for the powers of z^5:

$$
\begin{aligned}
0 &= \varepsilon - \frac{1}{2}(2\alpha\delta + 2\beta\gamma) + \left(\frac{1}{3}\right)(3\alpha^2\gamma + 3\alpha\beta^2) - \left(\frac{1}{4}\right)(4\alpha^3\beta) + \left(\frac{1}{5}\right)\alpha^5 \\
0 &= \varepsilon - \frac{1}{2}\left(2(1)\left(\frac{1}{24}\right) + 2\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)\right) + \left(\frac{1}{3}\right)\left(3\left(\frac{1}{6}\right) + 3\left(\frac{1}{2}\right)^2\right) - \left(\frac{1}{4}\right)\left(4\left(\frac{1}{2}\right)\right) + \left(\frac{1}{5}\right) \\
0 &= \varepsilon + \left(-\frac{1}{2}\right)\left(\frac{1}{12} + \frac{1}{6}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2} + \frac{3}{4}\right) - \left(\frac{1}{4}\right)(2) + \frac{1}{5} \\
0 &= \varepsilon + \left(-\frac{1}{2}\right)\left(\frac{1}{12} + \frac{2}{12}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{4} + \frac{3}{4}\right) - \frac{1}{2} + \frac{1}{5} \\
0 &= \varepsilon + \left(-\frac{1}{2}\right)\left(\frac{3}{12}\right) + \left(\frac{1}{3}\right)\left(\frac{5}{4}\right) - \frac{1}{2} + \frac{1}{5} \qquad (76) \\
0 &= \varepsilon - \frac{1}{8} + \frac{5}{12} - \frac{1}{2} + \frac{1}{5} \\
0 &= \varepsilon - \frac{15}{120} + \frac{50}{120} - \frac{60}{120} + \frac{24}{120} \\
0 &= \varepsilon - \frac{1}{120} \\
\varepsilon &= \frac{1}{120}
\end{aligned}
$$

Putting it all together:

$$
x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 + ... \qquad (77)
$$

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