

Newton's Power Series for Sine and Cosine

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In 1711, Isaac Newton (1642-1727) published a derivation of the power series for the sine in his seminal *Of Analysis by means of Equations with an infinite number of terms* [1, pp. 335-339, §37-§47]. This is the first appearance of the power series for the sine in a European Manuscript [2, p. 11]. It was possibly discovered as early as the mid-1660s¹.

We think that Newton's diagrams suffer from the use of too many letters, distracting the modern reader from the clarity of his argument. We will therefore use letters sparingly in our reconstruction, relying on colours instead. For the series inversion in §43-45, we will stick to Newton's method as closely as possible, but we also give a more modern treatment (the 'method of undetermined coefficients') in a subsection to §45.

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§37 How Fluents are generated from their Moments

In this section, Newton illustrates his claim that knowing the Moments of a quantity (or 'fluent') at all times allows you to calculate that quantity at all times as well. In function notation, knowing \dot{f} at all times allows you to find $f(t)$ by integration.

¹Newton writes, in the opening pages of the *Treatise on the Quadrature of Curves* that he discovered the inverse fluxional method in 1665-1666 :

I sought a Method of determining Quantities from the Velocities of the Motions or Increments, with which they are generated; and calling these Velocities of the Motions *Fluxions*, and the generated quantities *Fluents*, I fell by degrees upon the Method of Fluxions, which I have made use of here in the Quadrature of Curves, in the Years 1665 and 1666.

37. Let ABD be any Curve, and $AHKB$ a Rectangle, whose Side AH or BK is Unity : And imagine the Right Line DBK to move uniformly from AH , so as to describe the Areas ABD and AK ; and that BK (1) is the Moment with which AK (x), and BD (y) the Moment with which ABD is gradually encreased ;

$AHKB$ is a two-dimensional x -axis whose side length is 1 (see Figure 1). When considering areas under curves Newton prefers to consider a 2-dimensional x -axis, as opposed to a one-dimensional x -axis. But it is not mere convenience that makes Newton set the height of this rectangle (blue in Figure 1) equal to 1.

For our purposes it could be enlightening if we made the time-dependence of Newton's fluxional method explicit. Newton's 'uniform' motion can be satisfied by letting:

$$x = t \tag{1}$$

Then, $x = t \Rightarrow \dot{x} = 1$, and thus the Fluxion of x is 1 and the Moment BK is $1dt$ (or $1dx$). We come to this interpretation by reading an earlier section of Newton's *Treatise on the Quadrature of Curves*, in which Newton says that it is useful, sometimes, to take a quantity like x , set its first fluxion equal to 1 and all higher fluxions to 0. This implies $dx = dt$, so time-derivatives become x -derivatives, and we can therefore straightforwardly apply the 'inverse fluxional' method for geometrical integration problems:

But when one proceeds thus to second, third, and following fluxions it is proper to consider some Quantity as flowing uniformly, and for its first fluxion to write Unity, for the subsequent ones, nothing. and we can use the inverse fluxion method to integrate lengths.

— Isaac Newton, [3, §20, p. 7]

We interpret this particular case of a two-dimensional x -axis as a straightforward application of the principle Newton explains above. Newton continues:

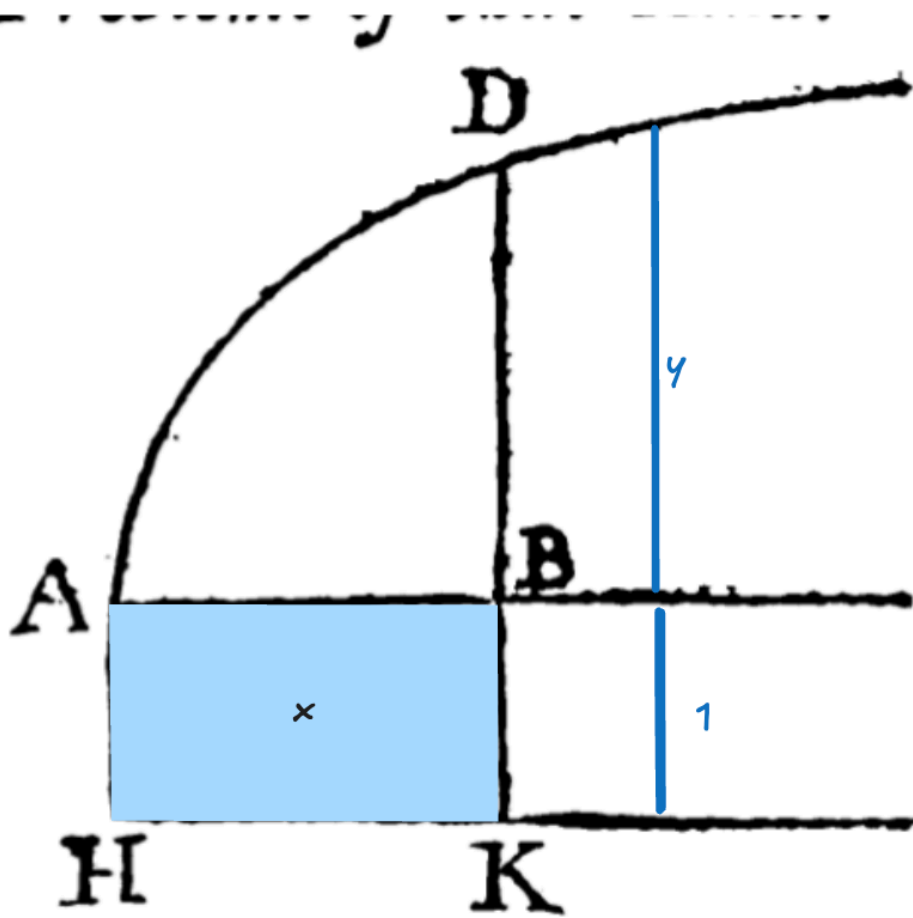


Figure 1: The blue area (x) is “increased continually by the Moment 1”.

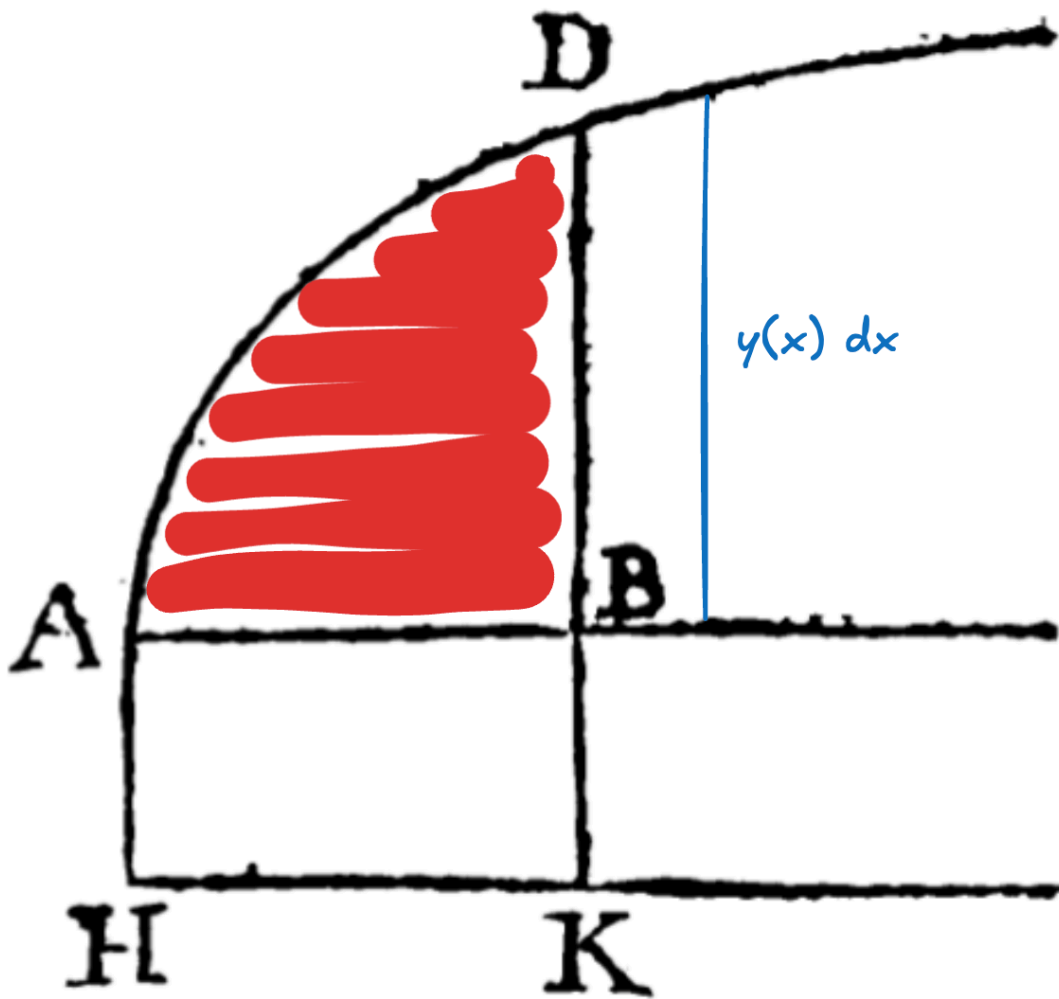


Figure 2: the red area is increased continually by the Moment $y(x)$. Equivalently, it is increased by the infinitesimal area segment $y(x)dx$

and that from the Moment BD continually given, you can, by Means of the preceding Rules, investigate the Area ABD described by it, or compare it with AK (x), which is described with the Moment r .



It is a puzzling to us that Newton calls BD and AK moments rather than fluxions, because we would recognize $y(x)$ and $\dot{x} = 1$ as derivatives of the blue (Figure 1) and red (Figure 2) areas respectively, rather than infinitesimal area segments².

This puzzle is not solved by finding Newton's earlier discussion of the Moment, in which he speaks of them as infinitesimal line segments (proportional to an infinitesimal quantity o):

²We follow Robert Pyke [4] in interpreting Fluents as time-varying quantities $y(t)$, Fluxions as their time-derivatives \dot{y} , and Moments as infinitesimal quantities ($\dot{y}dt$), or, in Newton's notation - o .

For let o be a very small Quantity, and let $o\dot{z}$, $o\dot{y}$, $o\dot{x}$ be the Moments, that is the momentaneous synchronal Increments of the Quantities z , y , x . And if the flowing Quantities are just now z , y , x , then after a Moment of Time, being increas'd by their Increments $o\dot{z}$, $o\dot{y}$, $o\dot{x}$, these Quantities shall become $z + o\dot{z}$, $y + o\dot{y}$, $x + o\dot{x}$

– [3, §17, p. 6]

In light of this quotation, it is especially puzzling that in this context, Newton writes 1 for the moment instead of $1o$.

Perhaps the reticence to explicitly use infinitesimal quantities in his treatment can be partly explained by the fact that Newton wants to avoid being seen to apply the Leibnizian second postulate³ of calculus, which he explicitly rejected in the opening of the *Analysis*:

I consider mathematical Quantities in this Place **not as consisting of very small Parts**; but as describ'd by a continued Motion. Lines are describ'd, and thereby generated not by the Apposition of Parts, but by the continued Motion of Points; Superficies's by the Motion of Lines; Solids by the Motion of Superficies'; Angles by the Rotation of the Sides; Portions of Time by a continual Flux: and so in other Quantities.

– [1, p. 1, §1], emphasis ours

§38 Finding the arc length of a circle in terms of a coordinate along its diameter.

In this section, Newton uses his method of *inverse fluxions* (integration) to find an arbitrary arc length of a circle in terms of a coordinate on its diameter. He first finds an expression of the tangent to the circle as a function of x as a power series, and then integrates it term by term to find the arc length.

³Any Curved line consists of infinitely many straight lines, each of which is infinitely small [5]

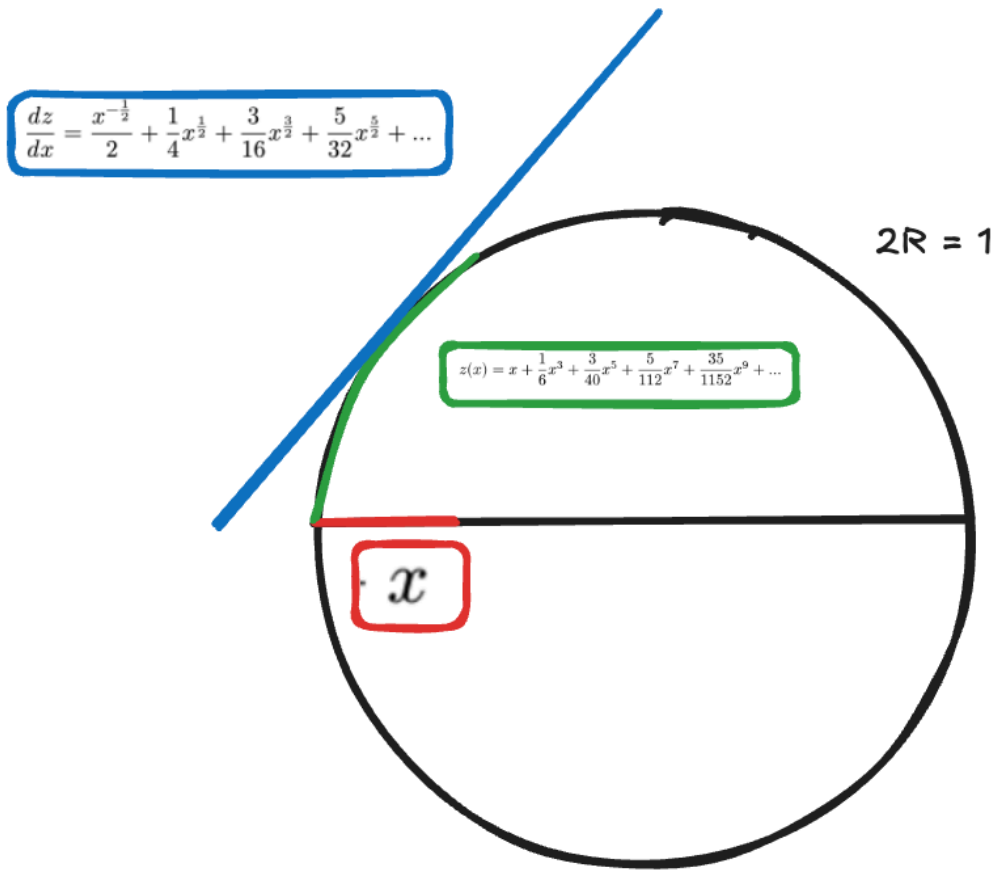


Figure 3: Summary of §38. Circle has diameter of 1

38. Let ADLE be a Circle, the Length of whose Arch AD is to be investigated. Draw the Tangent DHT, and having completed the indefinitely small Rectangle HGBK, and put $AE = 1 = 2AC$, it shall be as BK or GH the Moment of the Base AB (x) to HD the Moment of the Arch AD $:: BT : DT :: BD (\sqrt{x-xx}) : DC (\frac{1}{2}) :: 1 (BK) : \frac{1}{2\sqrt{x-xx}} (DH)$.

Interestingly, whereas §38 is presented as concrete example of §37, the dimensionalities are different. §37 is about integrating the ordinate $y(x)$ to find an area, whereas in §38 we integrate over infinitesimal line segments to find an arc length.

The double colon “ $::$ ” is just an equals sign. When Newton writes “ $BD (\sqrt{x-xx}) : DC (\frac{1}{2})$ ”, the brackets should not be taken to mean multiplication. Instead, they mean something akin to “in other words”. So, the statement, “ $A(B) : C(D)$ ” should be read as “ A (in other words, B) : C (in other words, D)”, or, equivalently $\frac{A}{C} = \frac{B}{D}$. For example, “ $1(BK) : \frac{1}{2\sqrt{x-xx}}(DH)$ ” actually

means “ $\frac{DH}{BK} = \frac{1}{2\sqrt{x-x^2}}$ ”. It is just a way to pack more equalities in the same line without writing an explicit equals sign.

Let us unpack Newton’s terse one-liner:

$$\frac{dz}{dx} \equiv \frac{DH}{GH} = \frac{DT}{BT} = \frac{DC}{DB} = \frac{1}{2\sqrt{x-x^2}} = \frac{DH}{BK} \quad (2)$$

Where does this expression come from? We start by noticing that the red and green triangles are similar (see Figure 4) since the two triangles share the angle $\angle BDT$ (or $\angle GDH$) and both triangles contain a right angle ($\angle TBD$ and $\angle HGD$), whence:

$$\frac{DT}{BT} = \frac{DH}{GH} \equiv \frac{dz}{dx} \quad (3)$$

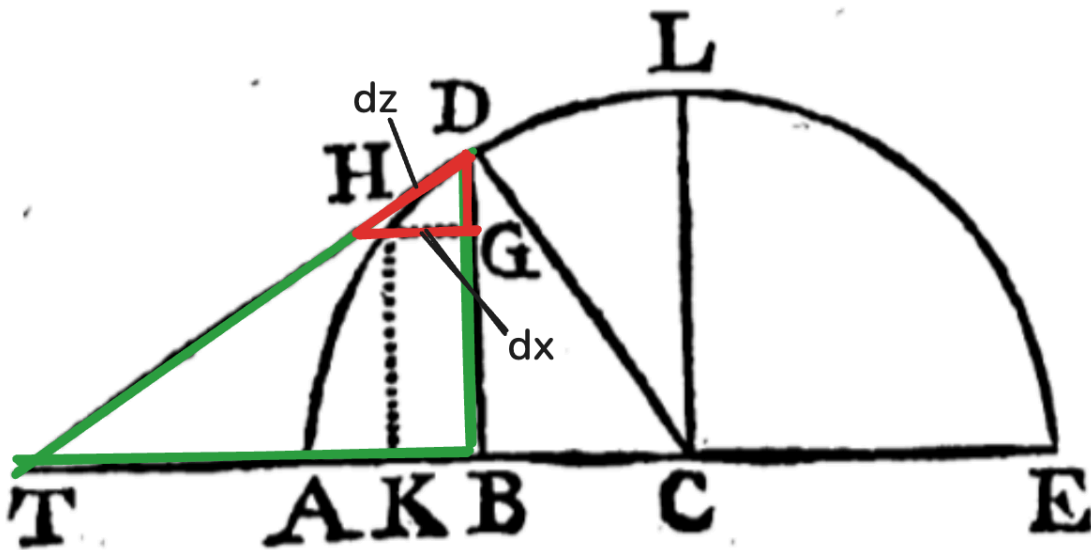


Figure 4: red and green triangles are similar

Next, we notice that the red and green triangles in Figure 5 are also similar, since both contain a right angle ($\angle TBD$ and $\angle DBC$) and $\angle CDB = \angle BTD$. The latter follows from the fact that in $\triangle DBT$, we can have $90^\circ - \angle BDT = \angle BTD$, and since $\angle CDT = 90^\circ$, we know that $\angle CDB = 90^\circ - \angle BDT$, which we established was equal to $\angle BTD$.

Therefore:

$$\frac{DT}{BT} = \frac{DC}{BD} \quad (4)$$

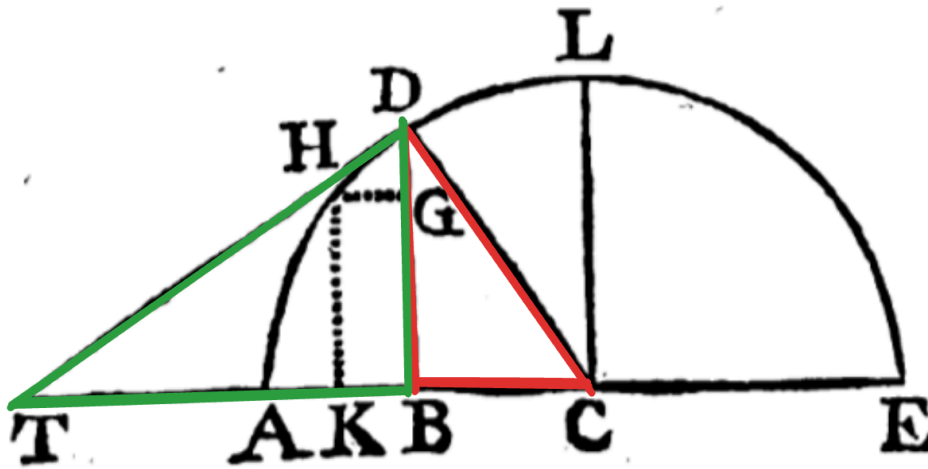


Figure 5: red and green triangles are similar

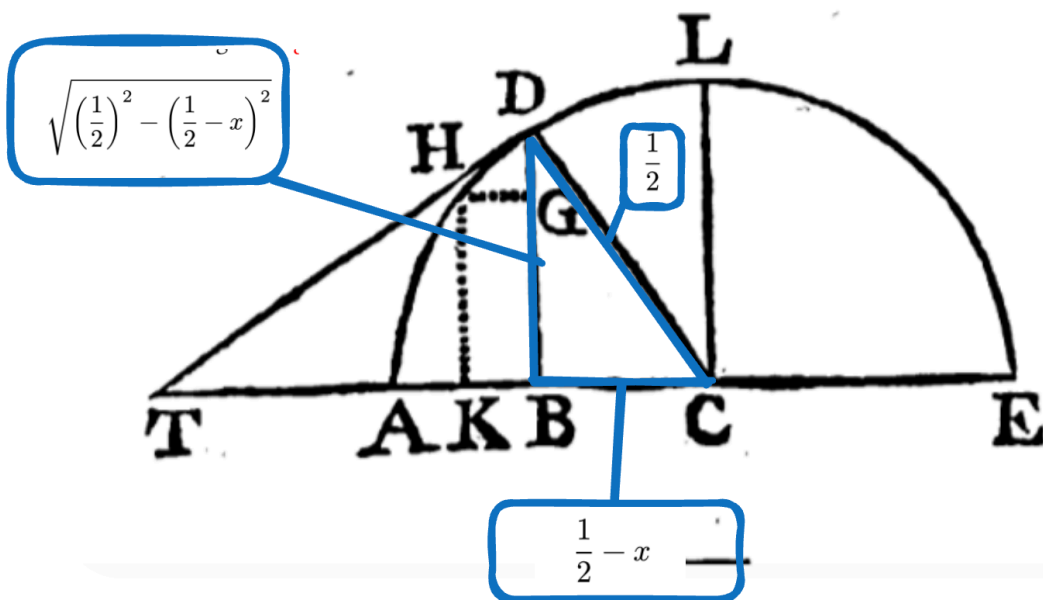


Figure 6: The Pythagorean theorem is used to find the value of the line BD
 Next, by using the Pythagorean theorem on the blue triangle in Figure 6, and the fact that

$$DC = \frac{1}{2}, \tag{5}$$

we find that:

$$BD = \sqrt{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2} - x\right)^2} = \sqrt{x - x^2} \quad (6)$$

Combining Equation 3 and Equation 4 to eliminate $\frac{DT}{BT}$ gives:

$$\frac{DH}{GH} = \frac{DC}{BD} \quad (7)$$

Using Equation 5 and Equation 6 to substitute for DC and BD gives:

$$\frac{dz}{dx} \equiv \frac{DH}{GH} = \frac{1}{2\sqrt{x - x^2}} \quad (8)$$

The arc length $z(x)$ can now be found by the inverse fluxional method (integration).

$$z(x) = \int \left(\frac{1}{2\sqrt{x - x^2}} \right) dx \quad (9)$$

In Newton's words, $\frac{1}{2\sqrt{x-x^2}}$ is the 'Moment' of the arc AD (or $z(x)$). Again, we are puzzled that he calls this the "Moment" rather than the "Fluxion" considering his own definitions for these terms. If this is really a Moment, where is the o ?

We can perform the integral by first writing out the series expansion for $\frac{1}{2\sqrt{x-x^2}}$ using the binomial theorem, and then integrating term-by-term.

The binomial theorem is easier to apply if we first rewrite the Moment as:

$$\frac{1}{2\sqrt{x - x^2}} = \frac{1}{2}(x - x^2)^{-\frac{1}{2}} = \frac{x^{-\frac{1}{2}}}{2}(1 - x)^{-\frac{1}{2}} \quad (10)$$

Next we expand $(1 - x)^{-\frac{1}{2}}$ using Newton's recursive version of the Binomial theorem [6], namely:

$$(P + PQ)^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \dots \quad (11)$$

Where $A = P^{\frac{m}{n}}$, $P = 1$ and $Q = -x$, $m = -1$ and $n = 2$, we have:

$$\begin{aligned} A &= 1 \\ B &= \frac{1}{2}x \\ C &= -\frac{3}{4}\left(\frac{1}{2}x\right)x = -\frac{3}{8}x^2 \\ D &= \frac{-1 - 2(2)}{3 \times 2}\left(\frac{3}{8}x^2\right)x = \frac{5}{16}x^3 \end{aligned} \quad (12)$$

So:

$$(1 - x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots \quad (13)$$

Therefore:

$$\frac{1}{2\sqrt{x - x^2}} = \frac{x^{-\frac{1}{2}}}{2}(1 - x)^{-\frac{1}{2}} = \frac{x^{-\frac{1}{2}}}{2} + \frac{1}{4}x^{\frac{1}{2}} + \frac{3}{16}x^{\frac{3}{2}} + \frac{5}{32}x^{\frac{5}{2}} + \dots \quad (14)$$

Now that we have obtained an expression for the moment of arc $\frac{dz}{dx}$ as a function of x (see Figure 7), we can integrate it term by term to get the arc length

$$\begin{aligned} \frac{dz}{dx} &= \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{4}x^{\frac{1}{2}} + \frac{3}{16}x^{\frac{3}{2}} + \frac{5}{32}x^{\frac{5}{2}} + \dots \\ z(x) &= \int z(x)dx = x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{3}{2}} + \frac{3}{40}x^{\frac{5}{2}} + \frac{5}{112}x^{\frac{7}{2}} \dots \end{aligned} \tag{15}$$

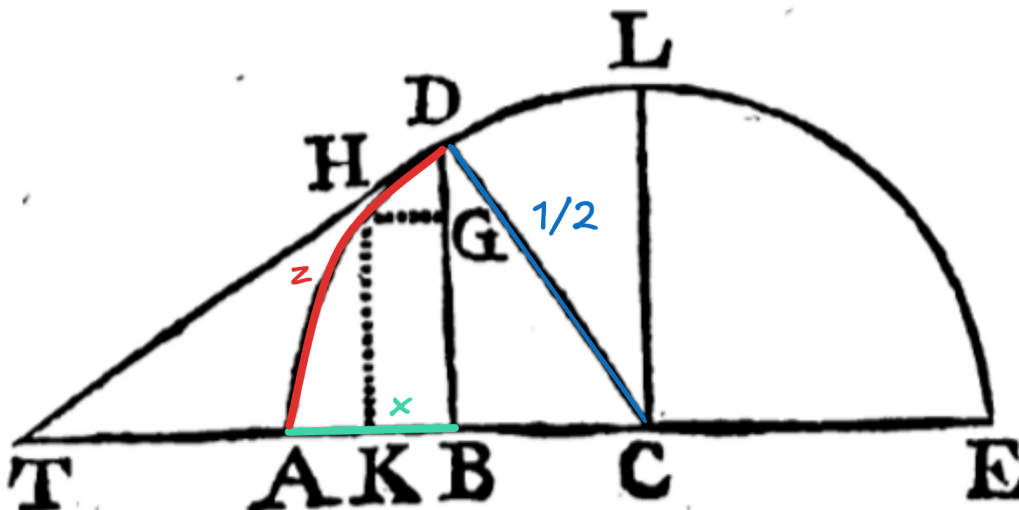


Figure 7: Finding $\frac{dz}{dx}$

This section also has an ambiguity regarding the use of the term Moment. First, the “Moment of the Arch AD” is supposed to be the infinitesimal line segment HD (Figure 1). Next, the *very same moment* is designated $\frac{\sqrt{x-x^2}}{2x-x^2}$, which looks a lot like an expression of the *derivative* of the arc length with respect to the x coordinate, $\frac{dz}{dx}$. If we are to interpret the Moment as a line segment, we would have to write $dz = \frac{\sqrt{x-x^2}}{2x-x^2}dx$, but Newton lacks the notation to do so.

§39 Finding the arc sine as a power series

In §38, Newton has shown that his method can find the arc length of a circle in terms of a coordinate along its diameter. But he has not yet found the *arcsine*. For that, he needs to consider a unit circle, and change his independent variable x to measure a distance from the center of the circle. (Compare Figure 8 with Figure 7 to see the difference in geometry between §39 and §38)

39. After the same Manner by supposing CB to be x , the Radius CA to be 1, you will find the Arch LD to be $x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c.

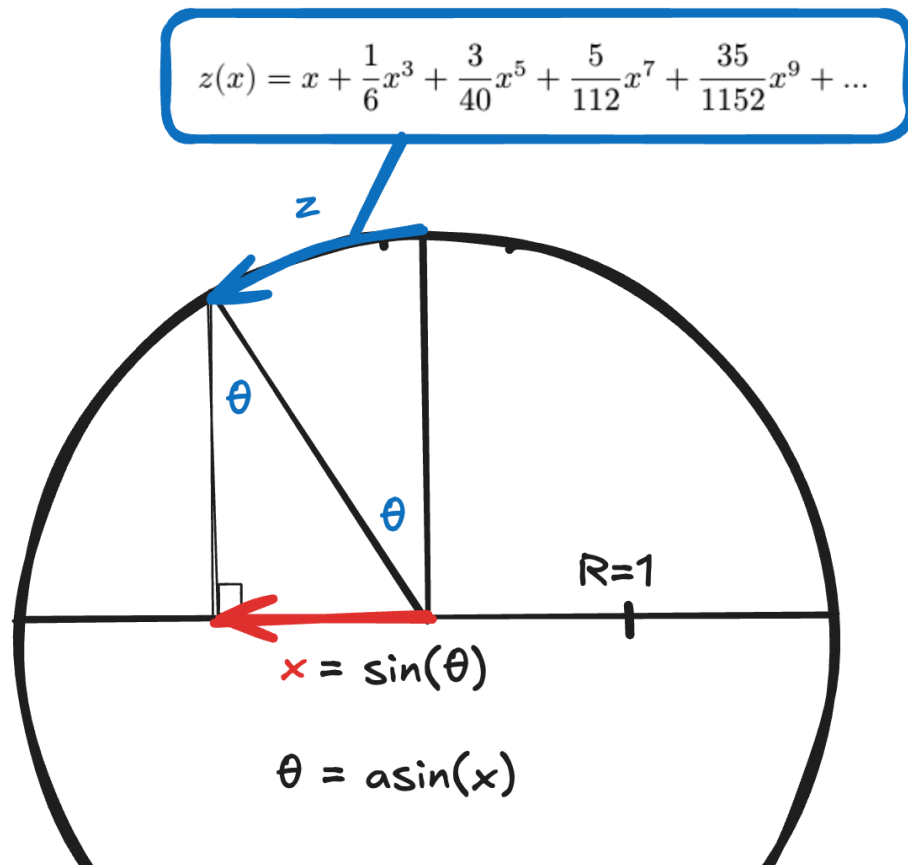


Figure 8: Summary of §39, compare with Figure 3

In contrast with §38, we now use a unit circle rather than a circle with diameter 1, so:

$$DC = 1 \tag{16}$$

The similar triangles argument that we used in §38 is still valid to establish the similarity between the infinitesimal triangle and DTB , as well as the similarity between DTB and DBC

But, instead of defining $AB = x$ we now have $BC = x$. By Pythagoras' theorem we obtain

$$BD = \sqrt{1 - x^2}. \tag{17}$$

The argument is repeated in Figure 9 without letters. This time we obtain a slightly different expression for the arc length (viz Equation 8):

$$\frac{dz}{dx} = \frac{1}{\sqrt{1 - x^2}} \tag{18}$$

which integrates to give:

$$z(x) = \int \left(\frac{1}{\sqrt{1 - x^2}} \right) dx. \tag{19}$$

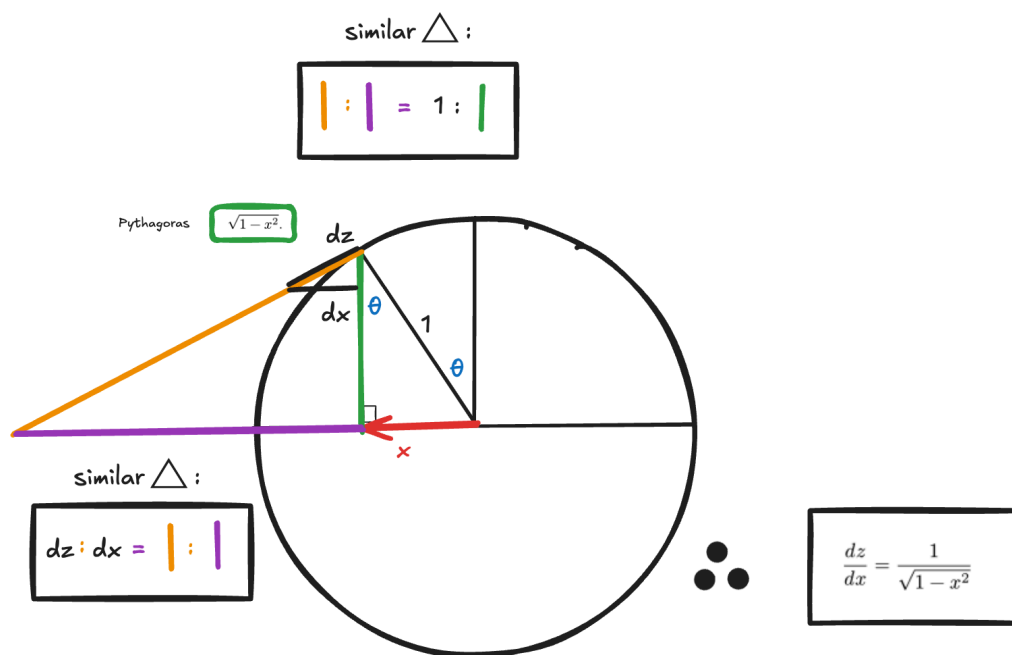


Figure 9: a similar triangles visualisation without letters.

As before, applying a series expansion to the integrand allows us to re-write this expression as

$$z(x) = \int \left(1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8 + \dots \right) dx \quad (20)$$

and integrating term by term gives

$$z(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \dots \quad (21)$$

Thus, we have obtained the power series expansion of the arcsine, which we will invert in section §43 to obtain the series expansion of the sine and cosine.

§40 An aside on the Dimensionality of Unity

40. But it is to be remarked that that Unity which is put for the Moment, is a Superficies, when the Question is about Solids; and a Line when about Superficies; and a Point when it is about Lines (as in this Example.) Neither am I afraid to speak of Unity in Points, or Lines infinitely small, since Geometers are wont now to consider Proportions even in such a Cafe, when they make use of the Methods of Indivisibles.

The discussion on the dimensionality of the Moment (as being one less than the dimensionality of the generated quantity), further casts into doubt whether it is fitting, as Robert Pyke and others have done [4], of interpreting of Moments as infinitesimal line (area, volume) segments. The derivative, however, does satisfy the property of being one dimension lower than the generated quantity.

§41 An aside on Solids and Centers of Gravity

41. From these Things one may guess how one ought to proceed in investigating the Superficies and Contents of Solids; and likewise the Centers of Gravity.

Regarding the contents, of solids, we remark that, if a 2D x axis is needed to allow us to compare the area $\int y(x)dx$ with x in §37, would we need a cylindrical x -axis with area 1 to allow us to compare the volume of rotation of a curve against x ?

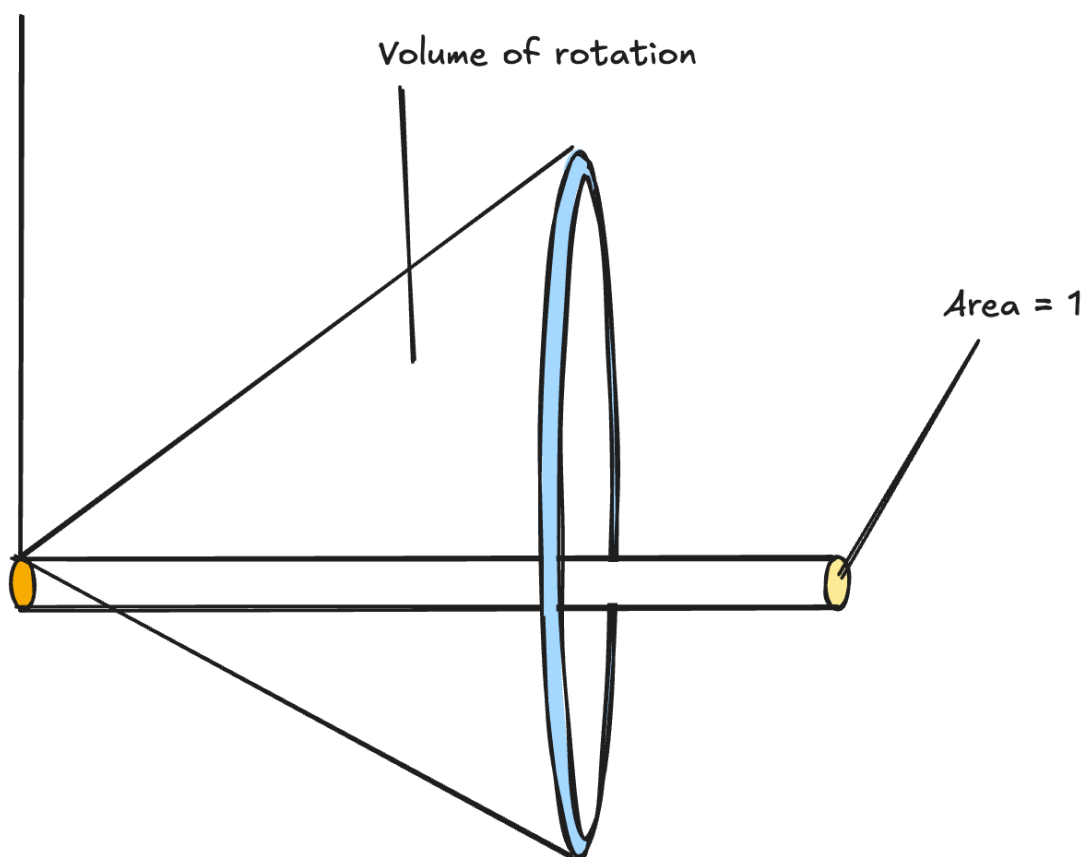


Figure 10: If Newton needs to construct a two-dimensional x -axis in order to rigorously compare how the area changes as a function of x , then would Newton have needed to construct a three-dimensional x -axis when considering a volume of rotation?

§42 Inverting the power series

To find the Converse of these Things.

42. But if upon the contrary, from the Area, or Length, &c. of any Curve being given, the Length of the Base AB be required, then you must extract the Root x , out of the Equations which have been found by the preceding Rules.

If we know the arc length z as a function of x , finding an expression of x as a function of z is equivalent to finding a series expansion of $\sin(z)$ in terms of z .

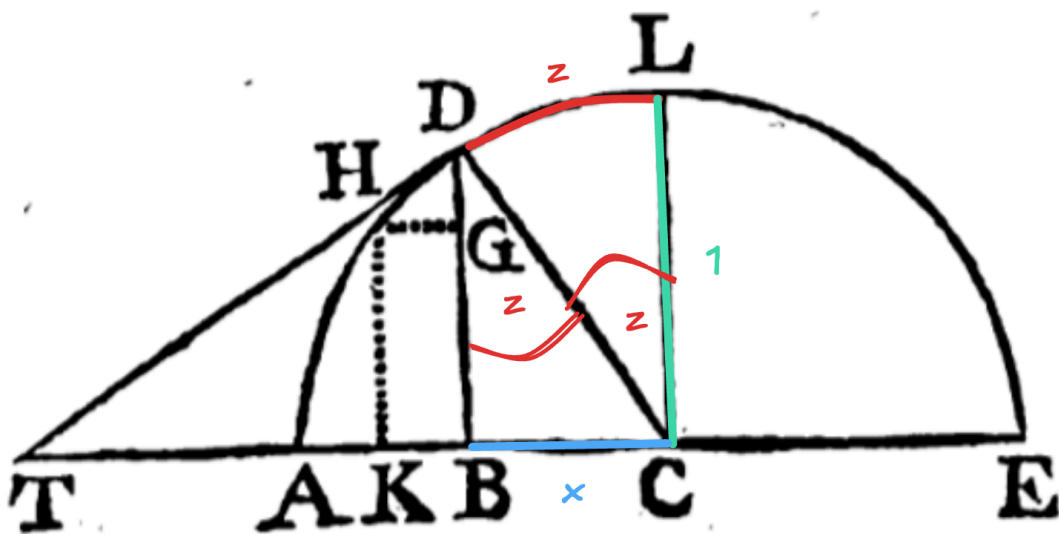


Figure 11: In a unit circle, the arc length z is also the angle z . Therefore, $x = \sin(z)$

Therefore, the expansion in Equation 47 is the expansion of the arcsine:

$$z(x) = \arcsin(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \dots \quad (22)$$

Inverting this expression will give us the series expansion of the sine:

$$x = \sin(z) \quad (23)$$

This will be the goal of section §43.

§43-44 Inverting $z = \ln(1 + x)$

Newton first demonstrates his series inversion method by inverting $z = \ln(1 + x)$ to get $x = e^z - 1$:

43. Thus if from the Area ABDC of the Hyperbola ($\frac{1}{1+x} = y$) given I wanted to investigate the Base AB, calling the Area z , I extract the Root of this Equation z (ABCD) $= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$, &c. neglecting those Terms in which x is of more Dimensions than z is desired in the Quotient.

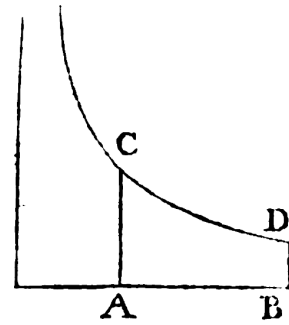


Figure 12: [7, p. 337]

It is no accident that he refers to x as a 'Quotient', since it appears to be the result of a process which closely resembles long-division, as we shall see.

So Newton starts with the power series:

$$\ln(1+x) \equiv z(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \dots \quad (24)$$

His aim is to find an expression like:

$$x(z) = \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 \dots \quad (25)$$

Newton is aware that if he wants to find coefficients of the inverse power series, up to z^5 all he needs to do is invert infinite series of $z(x)$ truncated after the term x^5 :

As if I would have z to rise to five Dimensions only in the Quotient, I neglect all the Terms $-\frac{1}{6}x^6 + \frac{1}{7}x^7 - \frac{1}{8}x^8$, &c. and extract the Root of this only $\frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - z = 0$.

Figure 13: [7, p. 337]

So, the infinite series:

$$z(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \dots \quad (26)$$

and the finite series:

$$z_5(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 \quad (27)$$

... will have a different inverse in general, but their inverses will be identical for the terms up to z^5 . Therefore, Newton can focus on inverting just the finite series $z_5(x)$.

Newton next rewrites Equation 27 and puts all terms on the LHS:

$$-z + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 = 0 \quad (28)$$

We see clearly from the expansion that:

$$z = x + O(x^2) + \dots \quad (29)$$

Therefore, the first-order approximation for z , which we call z_1 , is:

$$z_1 = x \quad (30)$$

The first-order approximation for x is given by inverting the first-order approximation for z . In other words,

$$x_1 = z \quad (31)$$

or

$$x = z + O(z^2) + \dots \quad (32)$$

Now, Newton rewrites Equation 28 by giving each power of x a separate row:

$$0 = \begin{cases} +\frac{1}{5}x^5 \\ -\frac{1}{4}x^4 \\ +\frac{1}{3}x^3 \\ -\frac{1}{2}x^2 \\ + x \\ - z \end{cases} \quad (33)$$

Now we make our substitution $x = z + p$, where p is taken to be $O(z^2)$

$$0 = \begin{cases} +\frac{1}{5}(z+p)^5 \\ -\frac{1}{4}(z+p)^4 \\ +\frac{1}{3}(z+p)^3 \\ -\frac{1}{2}(z+p)^2 \\ + z + p \\ - z \end{cases} \quad (34)$$

Then he expands Equation 34 selectively as follows:

$$0 = \begin{cases} +\frac{1}{5}z^5 + z^4p + \dots & 5 - 5 = 0 \\ -\frac{1}{4}z^4 - z^3p + \frac{3}{2}z^2p^2 + \dots & 5 - 4 = 1 \\ +\frac{1}{3}z^3 + z^2p + zp^2 + \frac{1}{3}zp^3 + \dots & 5 - 3 = 2 \\ -\frac{1}{2}z^2 - zp - \frac{1}{2}p^2 + 0 & 5 - 2 = 3 \\ +z^1 + p & \\ -z^1 & \end{cases} \quad (35)$$

We have kept the first ignored terms in grey, and point out that they are all of a higher order in z than z^5 . Newton uses the following rule to determine which terms to keep:

Rule; That after the first Term resulting from each Quantity that is collateral to it, I add no more Terms upon the right Hand than the Index of the Dimension of that first Term wants Units of the Index of the greatest Dimension. As in this Example, where the greatest Dimension is 5, I neglect all the Terms after z^5 , I put one after z^4 , and two only after z^3 . When the Root (x) to be extracted,

The Units of the Index of the greatest dimension is 5. The Index of the Dimension of the First term is 5 for the top row, and decreases by one for each row. Therefore $5 - 5 = 0$ and no additional terms are added after the z^5 term. The same logic leads us to conclude that one term is added after the z^4 term, and two after the z^3 term.

This means that terms like z^4p or z^2p^2 , which are $\sim O(z^6)$, will be dropped. These terms are irrelevant because we are only seeking to find the coefficients up to z^5 .

We are thus left with the following expression:

$$0 = \begin{cases} +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 - z^3p \\ +\frac{1}{3}z^3 + z^2p + zp^2 \\ -\frac{1}{2}z^2 - zp - \frac{1}{2}p^2 \\ +z + p \\ -z \end{cases} \quad (36)$$

Or, in Newton's original text:

$$\begin{array}{l} + \frac{1}{5}z^5 \text{ \Ô} \\ - \frac{1}{4}z^4 - z^3p \text{ \Ô} \\ + \frac{1}{3}z^3 + z^2p + zp^2 \text{ \Ô} \\ - \frac{1}{2}z^2 - zp - \frac{1}{2}p^2 \\ + z + p \\ - z \end{array}$$

Figure 14: [7, p. 337]

Since we are looking at the lowest term in the expansion for p , we find that Equation 36 can only hold if:

$$p = \frac{1}{2}z^2 + O(z^3) \quad (37)$$

So we have now found our second order expansion of $x(z)$:

$$x = z + \frac{z^2}{2} + O(z^3) \quad (38)$$

Newton's table up until this point looks like this:

$x + p = x$	$\begin{array}{r} + \frac{1}{5}x^5 \\ - \frac{1}{4}x^4 \\ + \frac{1}{3}x^3 \\ - \frac{1}{2}x^2 \\ + x \\ - x \end{array}$	$\begin{array}{r} + \frac{1}{5}z^5 \text{ \Ô} \\ - \frac{1}{4}z^4 - z^3p \text{ \Ô} \\ + \frac{1}{3}z^3 + z^2p + zp^2 \text{ \Ô} \\ - \frac{1}{2}z^2 - zp - \frac{1}{2}p^2 \\ + z + p \\ - z \end{array}$
$\frac{1}{2}z^2 + q = p$		

Figure 15: The steps to find the first and second terms in the inversion of $\ln(1+x)$

Now Newton returns to Equation 36, and rewrites it in order of decreasing dimensions of p and then z :

$$0 = \begin{cases} zp^2 \\ -\frac{1}{2}p^2 \\ -z^3p \\ +z^2p \\ -zp \\ +p \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases} \quad (39)$$

In Newton's original text, this looks like this:

$x + p = x$	$ \begin{array}{r} + \frac{1}{5}x^5 \\ - \frac{1}{4}x^4 \\ + \frac{1}{3}x^3 \\ - \frac{1}{2}x^2 \\ + x \\ - x \end{array} $	$ \begin{array}{r} + \frac{1}{5}z^5 \text{ \textcircled{c}} \\ - \frac{1}{4}z^4 - z^3p \text{ \textcircled{c}} \\ + \frac{1}{3}z^3 + z^2p + zp^2 \text{ \textcircled{c}} \\ - \frac{1}{2}z^2 - zp - \frac{1}{2}p^2 \\ + z + p \\ - z \end{array} $
$\frac{1}{2}z^2 + q = p$	$ \begin{array}{r} + zp^2 \\ - \frac{1}{2}p^2 \\ - z^3p \\ + z^2p \\ - zp \\ + p \\ + \frac{1}{5}z^5 \\ - \frac{1}{4}z^4 \\ + \frac{1}{3}z^3 \\ - \frac{1}{2}z^2 \end{array} $	

Figure 16: [7, p. 337]

Now, he substitutes $p = \frac{1}{2}z^2 + q$, where q is assumed to be of the form $q = O(z^3) + \dots$. He uses the same rules to determine which terms to keep

$$0 = \begin{cases} +z(\frac{1}{2}z^2 + q)^2 \\ -\frac{1}{2}(\frac{1}{2}z^2 + q)^2 \\ -z^3(\frac{1}{2}z^2 + q) \\ +z^2(\frac{1}{2}z^2 + q) \\ -z(\frac{1}{2}z^2 + q) \\ +(\frac{1}{2}z^2 + q) \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases} = \begin{cases} \frac{1}{4}z^5 + z^3q & 5 - 5 = 0 \\ -\frac{1}{8}z^4 - \frac{1}{2}z^2q & 5 - 4 = 1 \\ -\frac{1}{2}z^5 - z^3q & 5 - 5 = 0 \\ +\frac{1}{2}z^4 + z^2q & 5 - 4 = 1 \\ -\frac{1}{2}z^3 - qz & 5 - 3 = 2 \\ +\frac{1}{2}z^2 + q \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases} \quad (40)$$

Since q only enters linearly in Equation 40, we have a simple closed-form solution for q in terms of z :

$$x = z + \frac{1}{2}z^2 + q \quad (41)$$

where q is found by solving for q in Equation 40:

$$q = \frac{\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5}{1 - z + \frac{1}{2}z^2} \quad (42)$$

This can be found by polynomial long division, as Newton explains in §44.2:

2. When I see that p , q , or r , &c. in the last resulting Equation, is found of one Dimension only, I seek it's Value, that is to say the remaining Terms, which are still to be added to the Quotient, by means of Division; as you see done here.

Figure 17: [7, p. 338]

$$1 - z + \frac{1}{2}z^2 - \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5 \left(\frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 \right)$$

Figure 18: [7, p. 337]

We have recreated the long division below:

$$\begin{array}{r} 1 - z + \frac{1}{2}z^2 \quad \Bigg/ \quad \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5 \quad \Bigg\backslash \quad \frac{1}{6}z^3 \\ \hline \end{array}$$

Step 1: Dividing $\frac{1}{6}z^3$ by 1

$$\begin{array}{r} 1 - z + \frac{1}{2}z^2 \quad \Bigg/ \quad \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5 \quad \Bigg\backslash \quad \frac{1}{6}z^3 \\ \hline - \frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5 \\ \hline 0 + \frac{1}{24}z^4 + \frac{1}{30}z^5 \end{array}$$

Step 2: Calculating the remainder

$$\begin{array}{r} 1 - z + \frac{1}{2}z^2 \quad \Bigg/ \quad \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5 \quad \Bigg\backslash \quad \frac{1}{6}z^3 + \frac{1}{24}z^4 \\ \hline - \frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5 \\ \hline 0 + \frac{1}{24}z^4 - \frac{1}{30}z^5 \end{array}$$

Step 3: Dividing $\frac{1}{24}z^4$ by 1

$$\begin{array}{r} 1 - z + \frac{1}{2}z^2 \quad \Bigg/ \quad \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5 \quad \Bigg\backslash \quad \frac{1}{6}z^3 + \frac{1}{24}z^4 \\ \hline - \frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5 \\ \hline 0 + \frac{1}{24}z^4 - \frac{1}{30}z^5 \\ \hline - \frac{1}{24}z^4 + \frac{1}{24}z^5 \\ \hline 0 + \frac{1}{120}z^5 \end{array}$$

Step 4: Calculating the remainder

$$\begin{array}{r} 1 - z + \frac{1}{2}z^2 \quad \Bigg/ \quad \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5 \quad \Bigg\backslash \quad \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 \\ \hline - \frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5 \\ \hline 0 + \frac{1}{24}z^4 - \frac{1}{30}z^5 \\ \hline - \frac{1}{24}z^4 + \frac{1}{24}z^5 \\ \hline 0 + \frac{1}{120}z^5 \end{array}$$

Step 5: Dividing $\frac{1}{120}z^5$ by 1 to yield the final term

At this point we can stop, remembering that we sought the coefficients up to z^5 . We could, of course, continue the division beyond z^5 but these terms would be meaningless as the inverse of the infinite series $z(x)$ and the inverse of the finite series $z_5(x)$ cease to be identical after the z^5 term.

We have found q up to z^5 :

$$q = \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 + O(z^7) \quad (43)$$

We can plug this into our existing expansion for x :

$$x = z + \frac{1}{2}z^2 + q \quad (44)$$

Thus, the first 5 terms of the expansion of $e^z - 1$ is:

$$e^z - 1 \equiv x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 + \dots \quad (45)$$

If we wanted to find the next coefficient, we would have had to start with one more power of x in the series for z , namely:

$$z_6(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 \quad (46)$$

And use the same method.

This method is closely related to the modern method of undetermined coefficients, except the latter makes it more explicit that we are assuming an equation of the form $\sum_i^\infty C_i x^i$, whereas Newton's method leaves that implicit. We believe that the parallel with long division with decimal numbers would have made the explicit assumption of an equation of a certain form to be pedantic. After all, when dividing two decimal numbers, we do not start by *assuming* an answer of a certain form (with a decimal point and infinite digits after the decimal). We simply work it out. Similarly, Newton just works out the series without worrying about stating its form to begin with.

The completed table is shown below:

$x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{4}z^4 + \frac{1}{10}z^5 \text{ \Ô.}$		
$z + p = x$	$\begin{array}{r} + \frac{1}{2}z^2 \\ - \frac{1}{4}z^4 \\ + \frac{1}{3}z^3 \\ - \frac{1}{2}z^2 \\ + z \\ - z \end{array}$	$\begin{array}{r} + \frac{1}{2}z^2 \text{ \Ô.} \\ - \frac{1}{4}z^4 - z^3p \text{ \Ô.} \\ + \frac{1}{3}z^3 + z^2p + zp^2 \text{ \Ô.} \\ - \frac{1}{2}z^2 - zp - \frac{1}{2}p^2 \\ + z + p \\ - z \end{array}$
$\frac{1}{2}z^2 + q = p$	$\begin{array}{r} + zp^2 \\ - \frac{1}{2}p^2 \\ - z^3p \\ + z^2p \\ - zp \\ + p \\ + \frac{1}{2}z^5 \\ - \frac{1}{4}z^4 \\ + \frac{1}{3}z^3 \\ - \frac{1}{2}z^2 \end{array}$	$\begin{array}{r} + \frac{1}{4}z^5 \text{ \Ô.} \\ - \frac{1}{8}z^4 - \frac{1}{2}z^2q \\ - \frac{1}{2}z^5 \text{ \Ô.} \\ + \frac{1}{2}z^4 + z^2q \\ - \frac{1}{2}z^3 - zq \\ + \frac{1}{2}z^2 + q \\ + \frac{1}{5}z^5 \\ - \frac{1}{4}z^4 \\ + \frac{1}{3}z^3 \\ - \frac{1}{2}z^2 \end{array}$
$1 - z + \frac{1}{2}z^2 - \frac{1}{6}z^3 + \frac{1}{8}z^4 - \frac{1}{10}z^5 (\frac{1}{6}z^3 + \frac{1}{4}z^4 + \frac{1}{10}z^5)$		

Figure 19: Series inversion algorithm for $z = \ln(1 + x)$. Answer ($x = e^z - 1$)

§45 Inverting $z = \sin^{-1}(x)$ to find the sine.

Newton shows us his method in §43, and states without ceremony that we can apply it to his expression of the arcsine (Equation 47) to get a power series expression of the sine:

45. If from the Arch αD given the Sine AB was required; I extract the Root of the Equation found above, viz. $x = z + \frac{1}{6}z^3 + \frac{3}{40}z^5 + \frac{5}{112}z^7$ (it being supposed that $AB = x$, $\alpha D = z$, and $A\alpha = 1$) by which I find $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{30240}z^9 \text{ \Ô.}$

Figure 20: [7, p. 338]

Next, we will confirm that his method works. Let's explicitly apply what we learned in §43 to inverting $\sin^{-1}(x)$:

$$\sin^{-1}(x) \equiv z(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \dots \quad (47)$$

This series differs from $\ln(1+x)$ in that it contains only odd powers, and only positive coefficients. The coefficients themselves are more complicated, however. This time we want a solution up to the power of z^9 , so we must consider the truncated series:

$$z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 \quad (48)$$

The first order series is:

$$z = x + O(x^3) + \dots \quad (49)$$

Therefore, the inverse to first order is:

$$x = z + O(z^3) + \dots \quad (50)$$

As before, the first order inverse is $x_1 = z$. We find the next order by substituting $x = z + p$ into Equation 48. We rearrange it first into decreasing powers of x

$$0 = \begin{cases} +\frac{35}{1152}x^9 \\ +\frac{5}{112}x^7 \\ +\frac{3}{40}x^5 \\ +\frac{1}{6}x^3 \\ +x \\ -z \end{cases} \quad (51)$$

Now we substitute the expansion $x = z + p$

$$0 = \begin{cases} +\frac{35}{1152}(z+p)^9 \\ +\frac{5}{112}(z+p)^7 \\ +\frac{3}{40}(z+p)^5 \\ +\frac{1}{6}(z+p)^3 \\ +x \\ -z \end{cases} \quad (52)$$

Again, we don't need to expand all the terms, only the ones that will be of a lower order than x^9 , which we have done below.

$$0 = \begin{cases} +\frac{35}{1152}(z^9 + 9z^8p) + \dots & \frac{9-9}{2} = 0 \\ +\frac{5}{112}(z^7 + 7z^6p + 28z^5p^2) + \dots & \frac{9-7}{2} = 1 \\ +\frac{3}{40}(z^5 + 5z^4p + 10z^3p^2 + 10z^2p^3) + \dots & \frac{9-5}{2} = 2 \\ +\frac{1}{6}(z^3 + 3z^2p + 3zp^2 + p^3) + \dots & \frac{9-3}{2} = 3 \\ +z + p \\ -z \end{cases} \quad (53)$$

Again, we have added the next (ignored) term in grey. This step relies on the assumption that the lowest term in p will be $O(z^3)$. Today we might point out that $\sin^{-1}(x)$, having only odd terms in its power series expansion, is an odd function, and the inverse of an odd function is also odd.

Newton explains how we must modify his rule about which terms to keep for the case of an even or odd power series such as this one:

after z^4 , and two only after z^3 . When the Root (x) to be extracted, is every where of even or odd Dimensions, let this be the Rule: That after the first Term, resulting from each Quantity which is collateral to it, you add no more Terms towards the Right Hand, than what the Index of the Dimension of that first Term, wants Pairs of Units of the Index of the highest Dimension; or no more than what it wants Ternaries of Units, when the Indexes of the Dimensions of x differ by three Units; and so in others.

The “Index of the Highest Dimension” is 9. The “Index of the dimension of the first term” is 9 for the first row and drops by 2 in each row. “wants Pairs of Units” means we have to divide the difference by two to get the number of additional units on the right hand side.

To summarize, ignoring orders higher than $O(z^9)$ allows us to consider the (much simplified) Equation 54 rather than the full Equation 52

$$0 = \begin{cases} +\frac{35}{1152}z^9 + \dots \\ +\frac{5}{112}(z^7 + 7z^6p) + \dots \\ +\frac{3}{40}(z^5 + 5z^4p + 10z^3p^2) + \dots \\ +\frac{1}{6}(z^3 + 3z^2p + 3zp^2 + p^3) \\ +z + p \\ -z \end{cases} \quad (54)$$

It is clear that Equation 54 can only hold if:

$$p = -\frac{1}{6}z^3 + O(z^5) \quad (55)$$

We have thus found the third-order approximation:

$$x_3(z) = z - \frac{1}{6}z^3 \quad (56)$$

And thus we know the first two terms of our infinite power series expansion:

$$x(z) = z - \frac{1}{6}z^3 + \dots \quad (57)$$

We now rewrite Equation 53 in terms of decreasing powers of p and z , just like we did before (Equation 39) when calculating the inverse of $\ln(1+x)$ in §43. This arrangement makes it easier to substitute successive approximations for p , since terms of the same degree in p are grouped together.

$$0 = \begin{cases} +\frac{1}{6}p^3 \\ +\frac{3}{4}z^3p^2 \\ +\frac{1}{2}zp^2 \\ +\frac{35}{112}z^6p \\ +\frac{15}{40}z^4p \\ +\frac{1}{2}z^2p \\ p \\ +\frac{35}{1152}z^9 \\ +\frac{5}{112}z^7 \\ +\frac{3}{40}z^5 \\ +\frac{1}{6}z^3 \\ \cancel{+z} \\ \cancel{-z} \end{cases} \quad (58)$$

We now use the substitution $p = -\frac{1}{6}z^3 + q$ into Equation 58 where we assume that the leading coefficient of q is $O(z^5)$. Again, we use Newton's rule to discard terms of higher order than z^9 :

$$0 = \begin{cases} +\frac{1}{6}\left(-\frac{1}{6^3}z^9\right) + \dots & \frac{9-9}{2} = 0 \\ +\frac{3}{4}z^3\left(\frac{1}{36}z^6\right) + \dots & \frac{9-9}{2} = 0 \\ +\frac{1}{2}z\left(\frac{1}{36}z^6 \boxed{-\frac{1}{3}qz^3} + \dots\right) & \frac{9-7}{2} = \boxed{1} \\ +\frac{35}{112}z^6\left(-\frac{1}{6}z^3\right) + \dots & \frac{9-9}{2} = 0 \\ +\frac{15}{40}z^4\left(-\frac{1}{6}z^3 + \boxed{q}\right) & \frac{9-7}{2} = \boxed{1} \\ +\frac{1}{2}z^2\left(-\frac{1}{6}z^3 + q\right) & \frac{9-5}{2} = 2 \\ -\frac{1}{6}z^3 + q \\ +\frac{35}{1152}z^9 \\ +\frac{5}{112}z^7 \\ +\frac{3}{40}z^5 \\ +\frac{1}{6}z^3 \\ +z \\ -z \end{cases} \quad (59)$$

Expanding the brackets:

$$0 = \begin{cases} -\frac{1}{1296}z^9 + \dots \\ +\frac{1}{48}z^9 + \dots \\ +\frac{1}{72}z^7 & -\frac{1}{6}qz^4 + \dots \\ -\frac{35}{672}z^9 + \dots \\ -\frac{15}{240}z^7 & +\frac{15}{40}qz^4 + \dots \\ -\frac{1}{12}z^5 & +\frac{1}{2}z^2q \\ -\frac{1}{6}z^3 + q \\ +\frac{35}{1152}z^9 \\ +\frac{5}{112}z^7 \\ +\frac{3}{40}z^5 \\ +\frac{1}{6}z^3 \\ +z \\ -z \end{cases} \quad (60)$$

Adding like terms:

$$0 = \begin{cases} \left(-\frac{1}{1296} + \frac{1}{48} - \frac{35}{672} + \frac{35}{1152}\right)z^9 \\ \left(+\frac{1}{72} - \frac{15}{240} + \frac{5}{112}\right)z^7 \\ \left(+\frac{15}{40} - \frac{1}{6}\right)qz^4 \\ \left(-\frac{1}{12} + \frac{3}{40}\right)z^5 \\ +\frac{1}{2}z^2q \\ ~~-\frac{1}{6}z^3 + q~~ \\ ~~+\frac{1}{6}z^3~~ \end{cases} \quad (61)$$

At this point we find that our equation is linear in q , which means we can solve it using polynomial division:

$$0 = \begin{cases} -\frac{17}{10368}z^9 \\ -\frac{1}{252}z^7 \\ +\frac{5}{24}qz^4 \\ -\frac{1}{120}z^5 \\ +\frac{1}{2}z^2q \\ q \end{cases} \quad (62)$$

$$\frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 = q\left(1 + \frac{1}{2}z^2 + \frac{5}{24}z^4\right) \quad (63)$$

$$\frac{\frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9}{1 + \frac{1}{2}z^2 + \frac{5}{24}z^4} = q \quad (64)$$

We can calculate the remaining terms by straightforward polynomial division:

Step 1: Dividing $\frac{1}{120}z^5$ by 1

Step 2: Calculating the remainder

Step 3: Dividing $-\frac{1}{5040}z^7$ by 1

Step 4: Calculating the remainder

Step 5: Dividing $\frac{1}{362880}z^9$ by 1

We found that

$$q = \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 + O(z^{11}) \quad (65)$$

And therefore

$$\begin{aligned} x &= z - \frac{1}{6}z^3 + q \\ &= z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 + O(z^{11}) \end{aligned} \quad (66)$$

We recognize this as the first few terms of the series expansion of the sine:

$$x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \quad (67)$$

It is not clear from the manuscript whether Newton noticed the factorial pattern, as he doesn't use this notation.

This is the first appearance of the series expansion for the sine in any European Manuscript [2, p. 11].

We must point out that whereas we can continue the long division as long as we like, the resulting terms will have *nothing* to do with the sine. Instead they will be the continuation of the infinite series expansion of the inverse of the *truncated* expansion of the sine. To calculate the z^{13} we would have to start with the expansion of $z(x)$ up to order x^{13} and perform the same procedure.

Newton does not give us a table for his inversion of $\sin^{-1}(x)$ as he did for $\ln(1+x)$. But if he had, we can be fairly confident it would have looked something like this (viz. Figure 19):

$x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 + O(z^{11})$		
$x = z + p$	$ \begin{aligned} &+ \frac{35}{1152}x^9 \\ &+ \frac{5}{112}x^7 \\ &+ \frac{3}{40}x^5 \\ &+ \frac{1}{6}x^3 \\ &+ x \\ &- z \end{aligned} $	$ \begin{aligned} &+ \frac{35}{1152}z^9 \\ &+ \frac{5}{112}(z^7 + 7z^6p) \\ &+ \frac{3}{40}(z^5 + 5z^4p + 10z^3p^2) \\ &+ \frac{1}{6}(z^3 + 3z^2p + 3zp^2 + p^3) \\ &+ z + p \\ &- z \end{aligned} $
$p = -\frac{1}{6}z^3 + q$	$ \begin{aligned} &+ \frac{1}{6}p^3 \\ &+ \frac{3}{4}z^3p^2 \\ &+ \frac{1}{2}z^2p^2 \\ &+ \frac{35}{112}z^6p \\ &+ \frac{15}{40}z^4p \\ &+ \frac{1}{2}z^2p \\ &p \\ &+ \frac{35}{1152}z^9 \\ &+ \frac{5}{112}z^7 \\ &+ \frac{3}{40}z^5 \\ &+ \frac{1}{6}z^3 \\ &+ \\ &- z \end{aligned} $	$ \begin{aligned} &+ \frac{1}{6}\left(-\frac{1}{6^3}z^9\right) + \dots \\ &+ \frac{3}{4}z^3\left(\frac{1}{36}z^6\right) + \dots \\ &+ \frac{1}{2}z^2\left(\frac{1}{36}z^6 - \frac{1}{3}qz^3 + \dots\right) \\ &+ \frac{35}{112}z^6\left(-\frac{1}{6}z^3\right) + \dots \\ &+ \frac{15}{40}z^4\left(-\frac{1}{6}z^3 + q\right) \\ &+ \frac{1}{2}z^2\left(-\frac{1}{6}z^3 + q\right) \\ &-\frac{1}{6}z^3 + q \\ &+ \frac{35}{1152}z^9 \\ &+ \frac{5}{112}z^7 \\ &+ \frac{3}{40}z^5 \\ &+ \frac{1}{6}z^3 \\ &+ z \end{aligned} $
$ 1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 \quad \bigg/ \quad \frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 \quad \bigg/ \quad \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 $		

Using the ‘method of undetermined coefficients’

Newton’s method is computationally equivalent to the more modern “method of undetermined coefficients”, which is detailed below. We start with the $z(x)$ expression we had before:

$$z(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5.. \tag{68}$$

This time we explicitly assume that we can write x as a series expansion in terms of z :

$$x(z) = \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 \tag{69}$$

And we search for the coefficients term-by-term by repeatedly substituting Equation 69 into Equation 68. Our single equation turns into 5 equations in 5 variables, one equation for each power of z :

$$\begin{aligned}
 z(x) &= (\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5) \\
 &\quad - \frac{1}{2}(\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5)^2 \\
 &\quad + \frac{1}{3}(\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5)^3 \\
 &\quad - \frac{1}{4}(\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5)^4 \\
 &\quad + \frac{1}{5}(\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5)^5 \\
 &\quad \dots
 \end{aligned} \tag{70}$$

The powers of z^1 on LHS and RHS must be equal:

$$\alpha = 1 \tag{71}$$

The powers of z^2 on LHS and RHS must equal:

$$\begin{aligned}
 0 &= \beta + -\frac{1}{2}z^2 \\
 \beta &= \frac{1}{2}
 \end{aligned} \tag{72}$$

The same goes for the powers of z^3 :

$$\begin{aligned}
 0 &= \gamma + -\frac{1}{2}(2\beta\alpha) + \left(\frac{1}{3}\right)\alpha^3 \\
 \gamma &= \frac{1}{2} - \frac{1}{3} \\
 \gamma &= \frac{1}{6}
 \end{aligned} \tag{73}$$

The same goes for the powers of z^4 :

$$\begin{aligned}
0 &= \delta + \left(-\frac{1}{2}\right)(\beta^2 + 2\alpha\gamma) + \left(\frac{1}{3}\right)(3\alpha^2\beta) - \left(\frac{1}{4}\right)\alpha^4 \\
0 &= \delta + \left(-\frac{1}{2}\right)\left(\left(\frac{1}{2}\right)^2 + 2(1)\left(\frac{1}{6}\right)\right) + \left(\frac{1}{3}\right)\left(3(1)^2\left(\frac{1}{2}\right)\right) - \left(\frac{1}{4}\right)(1)^4 \\
0 &= \delta + \left(-\frac{1}{2}\right)\left(\frac{1}{4} + \frac{1}{3}\right) + \frac{1}{2} - \frac{1}{4} \\
0 &= \delta + \left(-\frac{1}{2}\right)\left(\frac{3}{12} + \frac{4}{12}\right) + \frac{12}{24} - \frac{6}{24} \\
0 &= \delta + \left(-\frac{1}{2}\right)\left(\frac{7}{12}\right) + \frac{12}{24} - \frac{6}{24} \\
0 &= \delta - \frac{7}{24} + \frac{12}{24} - \frac{6}{24} \\
0 &= \delta - \frac{1}{24} \\
\delta &= \frac{1}{24}
\end{aligned} \tag{74}$$

And finally, for the powers of z^5 :

$$\begin{aligned}
0 &= \varepsilon - \frac{1}{2}(2\alpha\delta + 2\beta\gamma) + \left(\frac{1}{3}\right)(3\alpha^2\gamma + 3\alpha\beta^2) - \left(\frac{1}{4}\right)(4\alpha^3\beta) + \left(\frac{1}{5}\right)\alpha^5 \\
0 &= \varepsilon - \frac{1}{2}\left(2(1)\left(\frac{1}{24}\right) + 2\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)\right) + \left(\frac{1}{3}\right)\left(3\left(\frac{1}{6}\right) + 3\left(\frac{1}{2}\right)^2\right) - \left(\frac{1}{4}\right)\left(4\left(\frac{1}{2}\right)\right) + \left(\frac{1}{5}\right) \\
0 &= \varepsilon + \left(-\frac{1}{2}\right)\left(\frac{1}{12} + \frac{1}{6}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2} + \frac{3}{4}\right) - \left(\frac{1}{4}\right)(2) + \frac{1}{5} \\
0 &= \varepsilon + \left(-\frac{1}{2}\right)\left(\frac{1}{12} + \frac{2}{12}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{4} + \frac{3}{4}\right) - \frac{1}{2} + \frac{1}{5} \\
0 &= \varepsilon + \left(-\frac{1}{2}\right)\left(\frac{3}{12}\right) + \left(\frac{1}{3}\right)\left(\frac{5}{4}\right) - \frac{1}{2} + \frac{1}{5} \\
0 &= \varepsilon - \frac{1}{8} + \frac{5}{12} - \frac{1}{2} + \frac{1}{5} \\
0 &= \varepsilon - \frac{15}{120} + \frac{50}{120} - \frac{60}{120} + \frac{24}{120} \\
0 &= \varepsilon - \frac{1}{120} \\
\varepsilon &= \frac{1}{120}
\end{aligned} \tag{75}$$

Putting it all together:

$$x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 + \dots \tag{76}$$

§46: Finding the power series expansion of the cosine

46. And moreover if the Cosine $A\beta$ were required from that Arch given, make $A\beta (= \sqrt{1 - xx}) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 - \frac{1}{362880}x^{10}, \&c.$

Finally, from the power series for the sine, Newton derives the power series for the cosine. He does this making use of the identity

$$\cos(z) = \sqrt{1 - \sin^2(x)}, \quad (77)$$

which follows from the fact that in Figure 22, $BD = \cos(z)$. We can then apply Pythagoras' theorem as follows:

$$BD = \sqrt{CD^2 - BC^2} = \sqrt{1 - x^2} \quad (78)$$

Substituting $x = \sin(z)$ gives:

$$BD = \sqrt{1 - \sin^2(x)} \quad (79)$$

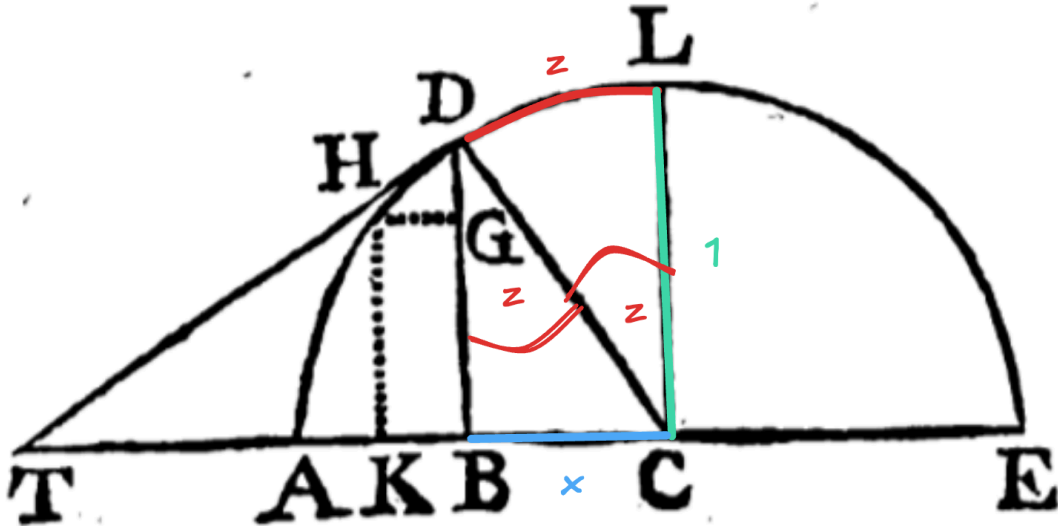


Figure 22: Repetition of Figure 11. From this figure it becomes apparent that the length $BD = \cos(z)$.

In order to find a power series for the cosine, we can therefore take the power series for $\sqrt{1 - x^2}$, and substitute for each x in this series the power series for $\sin(z)$. Therefore, our next step is to express $\sqrt{1 - x^2}$ as a power series. We can do this in the same manner we used for expanding $\frac{1}{BD} = \frac{1}{\sqrt{1 - x^2}}$ in section ... of this report, namely, by using Newton's recursive binomial formula. The power series thus obtained looks as follows:

$$(1 - x^2)^{-\frac{1}{2}} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 + \dots \quad (80)$$

Considering only the first three terms will suffice for our present purposes. The next step will be to plug our power series for the sine into each x in Equation 80. This gives:

$$\begin{aligned} \cos(z) = 1 - \frac{1}{2} \left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \dots \right)^2 - \frac{1}{8} \left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \dots \right)^4 \\ - \frac{1}{16} \left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \dots \right)^6 + \dots \end{aligned} \quad (81)$$

Now all we need to do is order the z terms. The first and second terms, in z^0 and z^2 , are easily found since they both appear only once. Therefore we have:

$$\cos(z) = 1 - \frac{1}{2}z^2 + \dots \quad (82)$$

The next term will be on the order of z^4 , so we need to collect all z^4 terms in the expansion. This gives us

$$\begin{aligned} \cos(z) = 1 - \frac{1}{2}z^2 - \frac{1}{2} \cdot -\frac{1}{6}z^3 \cdot z - \frac{1}{2} \cdot z \cdot -\frac{1}{6}z^3 - \frac{1}{8}z^4 + \dots \\ = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 + \dots \end{aligned} \quad (83)$$

Next, we collect all the z^6 terms, which gives us this:

$$\begin{aligned} \cos(z) = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - 2 \cdot \frac{1}{2}z \cdot \frac{1}{120}z^5 - \frac{1}{2} \cdot -\frac{1}{6}z^3 \cdot -\frac{1}{6}z^3 - 4 \cdot \frac{1}{8}z \cdot z \cdot z \cdot -\frac{1}{6}z^3 - \frac{1}{16}z^6 \\ = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6 + \dots \end{aligned} \quad (84)$$

Newton continues this procedure for two more terms, thus finding the coefficients for the z^8 and z^{10} terms as well.

§47: Recognizing the factorial pattern

The following section makes it clear that Newton saw the factorial pattern of the sine and cosine series, which today we would write as:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad (85)$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad (86)$$

Newton did not use these symbols as the sigma summation notation (\sum) was introduced by Euler in 1755 [8], and the factorial was introduced in 1808 by Christian Kramp [9].

Concerning the Continuation of the Series of the Progressions.

47. Let it be observed here, by the bye, that when 5 or 6 Terms of those Roots are known, they may be continued at Pleasure for most Part, by observing the Analogy of the Progression.

Thus you may continue this $x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$, &c. by dividing the last Term by the following Numbers in Order, 2, 3, 4, 5, 6, 7, &c.

And this $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7$, &c. by dividing by these Numbers 2×3 , 4×5 , 6×7 , 8×9 , 10×11 , &c.

And

Of the infinite Continuation of Terms.

And this $x = 1 - \frac{1}{2}z^2 + \frac{1}{4}z^4 - \frac{1}{720}z^6$, &c. by these 1×2 , 3×4 , 5×6 , 7×8 , 9×10 , &c.

And this $x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c. by multiplying by these, viz. $\frac{1 \times 1}{2 \times 3}$, $\frac{3 \times 3}{4 \times 5}$, $\frac{5 \times 5}{6 \times 7}$, $\frac{7 \times 7}{8 \times 9}$, &c. And so in others.

Concluding Remarks

Newton's derivation of the power series for sine and cosine is remarkable both for its ingenuity and its economy. Notably, Newton arrives at these results without Taylor series, which would not be formalized until decades later — he needed nothing more than elementary geometry, integration, the binomial theorem, and his method of series inversion. His recognition of the factorial pattern in §47 suggests he understood the deeper structure of the result, even without the notation to express it cleanly. That these discoveries were likely made in 1665-1666, when Newton was in his early twenties, makes them all the more striking.

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