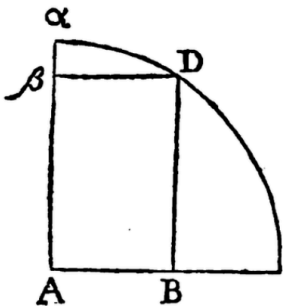


Newton's Power Series For Sine & Cosine (1669/1711)

June 9th, 2026

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45. If from the Arch αD given the Sine AB was required; I extract the Root of the Equation found above, viz. $x = x + \frac{1}{6}x^3 + \frac{1}{40}x^5 + \frac{5}{112}x^7$ (it being supposed that $AB = x$, $\alpha D = z$, and $A\alpha = 1$) by which I find $x = z - \frac{1}{6}z^3 +$

$$\frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 \text{ \&c.}$$

$\sin(x)$

46. And moreover if the Cofine AB were required from that Arch given, make $AB (=$

$$\sqrt{1 - xx}) = 1 - \frac{1}{2}z^2 + \frac{1}{4}z^4 - \frac{1}{720}z^6 + \frac{1}{40320}z^8 - \frac{1}{3628800}z^{10}, \text{ \&c.}$$

$\cos(x)$

Newton, Isaac, and John Stewart. *Sir Isaac Newton's Two Treatises of the Quadrature of Curves, And Analysis by Equations of an Infinite Number of Terms Explained*. Printed by James Bettenham for J. Nourse, 1745. https://books.google.it/books?id=noQ_AAAAcAAJ (p. 338)

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Finding a Power Series For $z = \sin^{-1}(x)$

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Inverting $z = \sin^{-1}(x)$ to find $x = \sin(z)$

Inverting $z = \ln(1 + x)$

43. Thus if from the Area ABDC of the Hyperbola ($\frac{1}{1+x} = y$) given I wanted to investigate the Base AB, calling the Area z , I extract the Root of this Equation $z(ABCD) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4, \&c.$ neglecting those Terms in which x is of more Dimensions than z is desired in the Quotient.

As if I would have a to rise to five Dimen-

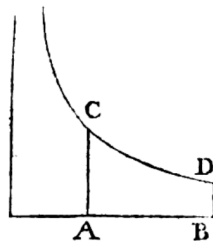
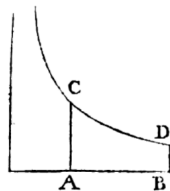


Figure 1: (Newton & Stewart, 1745, p. 337)

Goal:

$$x(z) = \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 \dots$$



$$\ln(1+x) \equiv z(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \dots$$

As if \bar{I} would have z to rise to five Dimensions only in the Quotient, I neglect all the Terms $-\frac{1}{8}x^6 + \frac{1}{7}x^7 - \frac{1}{4}x^8$, &c. and extract the Root of this only $\frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - z = 0$.

Figure 2: (Newton & Stewart, 1745, p. 337)

$$-z + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 = 0$$

$$0 = \begin{cases} +\frac{1}{5}x^5 \\ -\frac{1}{4}x^4 \\ +\frac{1}{3}x^3 \\ -\frac{1}{2}x^2 \\ + x \\ - z \end{cases}$$

$$x = z + p,$$

$$0 = \begin{cases} +\frac{1}{5}x^5 \\ -\frac{1}{4}x^4 \\ +\frac{1}{3}x^3 \\ -\frac{1}{2}x^2 \\ + x \\ - z \end{cases}$$

$$x = z + p,$$

$$0 = \begin{cases} +\frac{1}{5}(z+p)^5 \\ -\frac{1}{4}(z+p)^4 \\ +\frac{1}{3}(z+p)^3 \\ -\frac{1}{2}(z+p)^2 \\ + z + p \\ - z \end{cases}$$

$$0 = \left\{ \begin{array}{l} +\frac{1}{5}z^5 + z^4p + \dots \\ -\frac{1}{4}z^4 - z^3p + \frac{3}{2}z^2p^2 + \dots \\ +\frac{1}{3}z^3 + z^2p + zp^2 + \frac{1}{3}zp^3 + \dots \\ -\frac{1}{2}z^2 - zp - \frac{1}{2}p^2 + 0 \\ \cancel{+z^1} + p \\ \cancel{-z^1} \end{array} \right. \quad \begin{array}{l} 5 - 5 = 0 \\ 5 - 4 = 1 \\ 5 - 3 = 2 \\ 5 - 2 = 3 \end{array}$$

5,4,3 ...

5

after the first Term resulting from each Quantity that is collateral to it, I add no more Terms upon the right Hand than the Index of the Dimension of that first Term wants Units of the Index of the greatest Dimension.

$$0 = \begin{cases} +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 - z^3p \\ +\frac{1}{3}z^3 + z^2p + zp^2 \\ -\frac{1}{2}z^2 - zp - \frac{1}{2}p^2 \\ +z + p \\ -z \end{cases}$$

$$\begin{array}{r}
+ \frac{1}{5}z^5 \text{ etc.} \\
- \frac{1}{4}z^4 - z^3p \text{ etc.} \\
+ \frac{1}{3}z^3 + z^2p + zp^2 \text{ etc.} \\
- \frac{1}{2}z^2 - zp - \frac{1}{2}p^2 \\
+ z + p \\
- z
\end{array}$$

Figure 3: (Newton & Stewart, 1745, p. 337)

$$p = \frac{1}{2}z^2 + O(z^3)$$

$$\begin{aligned}
 &+ \frac{1}{5}z^5 \text{ \Öc.} \\
 &- \frac{1}{4}z^4 - z^3p \text{ \Öc.} \\
 &+ \frac{1}{3}z^3 + z^2p + zp^2 \text{ \Öc.} \\
 &- \frac{1}{2}z^2 - zp - \frac{1}{2}p^2 \\
 &+ z + p \\
 &- z
 \end{aligned}$$

Figure 3: (Newton & Stewart, 1745, p. 337)

$z + p = x$	$ \begin{aligned} &+ \frac{1}{5}x^5 \\ &- \frac{1}{4}x^4 \\ &+ \frac{1}{3}x^3 \\ &- \frac{1}{2}x^2 \\ &+ x \\ &- z \end{aligned} $	$ \begin{aligned} &+ \frac{1}{5}z^5 \text{ } \mathcal{O}c. \\ &- \frac{1}{4}z^4 - z^3p \text{ } \mathcal{O}c. \\ &+ \frac{1}{3}z^3 + z^2p + zp^2 \text{ } \mathcal{O}c. \\ &- \frac{1}{2}z^2 - zp - \frac{1}{2}p^2 \\ &+ z + p \\ &- z \end{aligned} $
$\frac{1}{2}z^2 + q = p$		

$$z + p = x$$

$$+ \frac{1}{5}x^5$$

$$- \frac{1}{4}x^4$$

$$+ \frac{1}{3}x^3$$

$$- \frac{1}{2}x^2$$

$$+ x$$

$$- z$$

$$+ \frac{1}{5}z^5 \text{ \Ô.}$$

$$- \frac{1}{4}z^4 - z^3p \text{ \Ô.}$$

$$+ \frac{1}{3}z^3 + z^2p + zp^2 \text{ \Ô.}$$

$$- \frac{1}{2}z^2 - zp - \frac{1}{2}p^2$$

$$+ z + p$$

$$- z$$

$$\frac{1}{2}z^2 + q = p$$

$$0 = \begin{cases} +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 - z^3p \\ +\frac{1}{3}z^3 + z^2p + zp^2 \\ -\frac{1}{2}z^2 - zp - \frac{1}{2}p^2 \\ +z + p \\ -z \end{cases}$$



$$0 = \begin{cases} zp^2 \\ -\frac{1}{2}p^2 \\ -z^3p \\ +z^2p \\ -zp \\ +p \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases}$$

$x + p = x$	$+$ $\frac{1}{2}x^5$ $-$ $\frac{1}{4}x^4$ $+$ $\frac{1}{6}x^3$ $-$ $\frac{1}{8}x^2$ $+$ x $-$ x	$+$ $\frac{1}{2}x^5$ Et c. $-$ $\frac{1}{4}x^4 - x^3p$ Et c. $+$ $\frac{1}{6}x^3 + x^2p + xp^2$ Et c. $-$ $\frac{1}{8}x^2 - xp - \frac{1}{2}p^2$ $+$ $x + p$ $-$ x
$\frac{1}{2}x^2 + q = p$	$+$ xp^2 $-$ $\frac{1}{2}p^2$ $-$ x^3p $+$ x^2p $-$ xp $+$ p $+$ $\frac{1}{5}x^5$ $-$ $\frac{1}{4}x^4$ $+$ $\frac{1}{3}x^3$ $-$ $\frac{1}{2}x^2$	

Figure 4: (Newton & Stewart, 1745, p. 337)

$$0 = \begin{cases} +z\left(\frac{1}{2}z^2 + q\right)^2 \\ -\frac{1}{2}\left(\frac{1}{2}z^2 + q\right)^2 \\ -z^3\left(\frac{1}{2}z^2 + q\right) \\ +z^2\left(\frac{1}{2}z^2 + q\right) \\ -z\left(\frac{1}{2}z^2 + q\right) \\ +\left(\frac{1}{2}z^2 + q\right) \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases} = \begin{cases} \frac{1}{4}z^5 + z^3q \\ -\frac{1}{8}z^4 - \frac{1}{2}z^2q \\ -\frac{1}{2}z^5 - z^3q \\ +\frac{1}{2}z^4 + z^2q \\ -\frac{1}{2}z^3 - qz \\ +\frac{1}{2}z^2 + q \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases} \quad \begin{array}{l} 5 - 5 = 0 \\ 5 - 4 = 1 \\ 5 - 5 = 0 \\ 5 - 4 = 1 \\ 5 - 3 = 2 \end{array}$$

$$0 = \begin{cases} +z\left(\frac{1}{2}z^2 + q\right)^2 \\ -\frac{1}{2}\left(\frac{1}{2}z^2 + q\right)^2 \\ -z^3\left(\frac{1}{2}z^2 + q\right) \\ +z^2\left(\frac{1}{2}z^2 + q\right) \\ -z\left(\frac{1}{2}z^2 + q\right) \\ +\left(\frac{1}{2}z^2 + q\right) \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases} = \begin{cases} \frac{1}{4}z^5 + z^3q \\ -\frac{1}{8}z^4 - \frac{1}{2}z^2q \\ -\frac{1}{2}z^5 - z^3q \\ +\frac{1}{2}z^4 + z^2q \\ -\frac{1}{2}z^3 - qz \\ +\frac{1}{2}z^2 + q \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases} \begin{matrix} 5 - 5 = 0 \\ 5 - 4 = 1 \\ 5 - 5 = 0 \\ 5 - 4 = 1 \\ 5 - 3 = 2 \end{matrix}$$

LINEAR in q

$$q = \frac{\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5}{1 - z + \frac{1}{2}z^2}$$

2. When I see that p , q , or r , &c. in the last resulting Equation, is found of one Dimension only, I seek it's Value, that is to say the remaining Terms, which are still to be added to the Quotient, by means of Division; as you see done here.

Figure 5: (Newton & Stewart, 1745, p. 338)

$$1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 \left(\frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 \right)$$

Figure 6: (Newton & Stewart, 1745, p. 337)

$$q = \frac{\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5}{1 - z + \frac{1}{2}z^2}$$

$$\begin{array}{r} \textcircled{1} - z + \frac{1}{2}z^2 \overline{) \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5} \\ \underline{z^3} \\ z^4 - \frac{1}{2}z^5 \\ z^5 - \frac{1}{4}z^6 \\ z^6 - \frac{1}{2}z^7 \end{array}$$

$$\begin{array}{r}
 \left(1 - z + \frac{1}{2}z^2\right) \left(\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5\right) \\
 \hline
 \left(-\frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5\right) \\
 \hline
 0 + \frac{1}{24}z^4 + \frac{1}{30}z^5
 \end{array}$$

The diagram shows a multiplication of two power series. The first series is $1 - z + \frac{1}{2}z^2$ and the second is $\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5$. The result of the multiplication is $-\frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5$. A horizontal line is drawn below this result, and the final simplified result is $0 + \frac{1}{24}z^4 + \frac{1}{30}z^5$. Blue circles and arrows highlight the terms and their relationships: the first series is circled, and arrows point from it to the intermediate result and the final result. The intermediate result is also circled, and an arrow points from it to the final result. The term $\frac{1}{6}z^3$ in the second series is circled, and an arrow points from it to the $-\frac{1}{6}z^3$ term in the intermediate result.

$$\begin{array}{r}
 \textcircled{1} - z + \frac{1}{2}z^2 \quad / \quad \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5 \quad \backslash \quad \frac{1}{6}z^3 + \textcircled{\frac{1}{24}z^4} \\
 - \frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5 \\
 \hline
 0 \quad \textcircled{+\frac{1}{24}z^4} - \frac{1}{30}z^5
 \end{array}$$

A blue arrow points from the circled $\frac{1}{24}z^4$ term in the remainder to the circled 1 in the dividend, with a blue percentage symbol (%) next to it.

$$\begin{array}{r}
 \left(1 - z + \frac{1}{2}z^2\right) \times \left(\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5\right) \\
 \hline
 \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5 \\
 - \frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5 \\
 \hline
 0 + \frac{1}{24}z^4 - \frac{1}{30}z^5 \\
 \hline
 -\frac{1}{24}z^4 + \frac{1}{24}z^5 \\
 \hline
 0 + \frac{1}{120}z^5
 \end{array}$$

The diagram shows a long division process. The dividend is $\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5$ and the divisor is $1 - z + \frac{1}{2}z^2$. The quotient is $\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5$. The remainder is $-\frac{1}{24}z^4 + \frac{1}{24}z^5$. The final result is $0 + \frac{1}{120}z^5$. Blue circles highlight the divisor, the dividend, the quotient, and the remainder. Blue arrows indicate the flow of the calculation.

$$\begin{array}{r}
 \textcircled{1} - z + \frac{1}{2}z^2 \quad \Bigg/ \quad \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5 \quad \Bigg/ \quad \frac{1}{6}z^3 + \frac{1}{24}z^4 + \textcircled{\frac{1}{120}z^5} \\
 \hline
 - \frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5 \\
 \hline
 0 + \frac{1}{24}z^4 - \frac{1}{30}z^5 \\
 - \frac{1}{24}z^4 + \frac{1}{24}z^5 \\
 \hline
 0 + \textcircled{\frac{1}{120}z^5}
 \end{array}$$

%

$$q = \frac{\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5}{1 - z + \frac{1}{2}z^2}$$

Long Division

$$q = \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 + O(z^7)$$

Long division:

$$q = \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 + O(z^7)$$

$$x = z + \frac{1}{2}z^2 + q$$

$$e^z - 1 \equiv x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 + \dots$$

$x = x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 \text{ \&c.}$		
$x + p = x$	$+$ $\frac{1}{2}x^5$ $-$ $\frac{1}{4}x^4$ $+$ $\frac{1}{2}x^3$ $-$ $\frac{1}{2}x^2$ $+$ x $-$ x	$+$ $\frac{1}{2}x^5 \text{ \&c.}$ $-$ $\frac{1}{4}x^4 - x^3p \text{ \&c.}$ $+$ $\frac{1}{2}x^3 + x^2p + xp^2 \text{ \&c.}$ $-$ $\frac{1}{2}x^2 - xp - \frac{1}{2}p^2$ $+$ $x + p$ $-$ x
$\frac{1}{2}x^2 + q = p$	$+$ xp^2 $-$ $\frac{1}{2}p^3$ $-$ x^3p $+$ x^2p $-$ xp $+$ p $+$ $\frac{1}{2}x^5$ $-$ $\frac{1}{4}x^4$ $+$ $\frac{1}{2}x^3$ $-$ $\frac{1}{2}x^2$	$+$ $\frac{1}{2}x^5 \text{ \&c.}$ $-$ $\frac{1}{4}x^4 - \frac{1}{2}x^2q$ $-$ $\frac{1}{2}x^5 \text{ \&c.}$ $+$ $\frac{1}{2}x^4 + x^2q$ $-$ $\frac{1}{2}x^3 - xq$ $+$ $\frac{1}{2}x^2 + q$ $+$ $\frac{1}{2}x^5$ $-$ $\frac{1}{4}x^4$ $+$ $\frac{1}{2}x^3$ $-$ $\frac{1}{2}x^2$
$1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 (\frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$		

Figure 19: (Newton & Stewart, 1745, p. 337)

Inverting $z = \sin^{-1}(x)$.

infinite

$$\sin^{-1}(x) \equiv z(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \dots$$

finite

$$z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9$$

$$z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9$$

rearrange:

$$x = z + p$$

$$0 = \left\{ \begin{array}{l} + \frac{35}{1152}x^9 \\ + \frac{5}{112}x^7 \\ + \frac{3}{40}x^5 \\ + \frac{1}{6}x^3 \\ + x \\ - z \end{array} \right.$$

substitute

$$x = z + p$$

$$0 = \begin{cases} +\frac{35}{1152}(z+p)^9 \\ +\frac{5}{112}(z+p)^7 \\ +\frac{3}{40}(z+p)^5 \\ +\frac{1}{6}(z+p)^3 \\ +x \\ -z \end{cases}$$

$$0 = \begin{cases} +\frac{35}{1152}(z^9 + 9z^8p) + \dots & \frac{9-9}{2} = 0 \\ +\frac{5}{112}(z^7 + 7z^6p + 28z^5p^2) + \dots & \frac{9-7}{2} = 1 \\ +\frac{3}{40}(z^5 + 5z^4p + 10z^3p^2 + 10z^2p^3) + \dots & \frac{9-5}{2} = 2 \\ +\frac{1}{6}(z^3 + 3z^2p + 3zp^2 + p^3) + \dots & \frac{9-3}{2} = 3 \\ +z + p \\ -z \end{cases}$$

9, 7, 5, 3...

9

9/2

after the first Term, resulting from each Quantity which is collateral to it, you add no more Terms towards the Right Hand, than what the Index of the Dimension of that first Term, wants Pairs of Units of the Index of the highest Dimension;

$$0 = \left\{ \begin{array}{l} + \frac{35}{1152} z^9 \\ + \frac{5}{112} (z^7 + 7z^6 p) \\ + \frac{3}{40} (z^5 + 5z^4 p + 10z^3 p^2) \\ + \frac{1}{6} (z^3 + 3z^2 p + 3z p^2 + p^3) \\ + z \quad + p \\ - z \end{array} \right.$$

$$p = -\frac{1}{6}z^3 + O(z^5)$$

$$0 = \begin{cases} +\frac{35}{1152}z^9 \\ +\frac{5}{112}(z^7 + 7z^6p) \\ +\frac{3}{40}(z^5 + 5z^4p + 10z^3p^2) \\ +\frac{1}{6}(z^3 + 3z^2p + 3zp^2 + p^3) \\ +z + p \\ -z \end{cases}$$

rearrange

$$0 = \begin{cases} +\frac{35}{1152}z^9 \\ +\frac{5}{112}(z^7 + 7z^6p) \\ +\frac{3}{40}(z^5 + 5z^4p + 10z^3p^2) \\ +\frac{1}{6}(z^3 + 3z^2p + 3zp^2 + p^3) \\ +z + p \\ -z \end{cases}$$



$$0 = \begin{cases} +\frac{1}{6}p^3 \\ +\frac{3}{4}z^3p^2 \\ +\frac{1}{2}zp^2 \\ +\frac{35}{112}z^6p \\ +\frac{15}{40}z^4p \\ +\frac{1}{2}z^2p \\ p \\ +\frac{35}{1152}z^9 \\ +\frac{5}{112}z^7 \\ +\frac{3}{40}z^5 \\ +\frac{1}{6}z^3 \\ \cancel{+z} \\ \cancel{-z} \end{cases}$$

substitute

$$0 = \begin{cases} +\frac{1}{6}p^3 \\ +\frac{3}{4}z^3p^2 \\ +\frac{1}{2}zp^2 \\ +\frac{35}{112}z^6p \\ +\frac{15}{40}z^4p \\ +\frac{1}{2}z^2p \\ p \\ +\frac{35}{1152}z^9 \\ +\frac{5}{112}z^7 \\ +\frac{3}{40}z^5 \\ +\frac{1}{6}z^3 \\ \cancel{z} \\ \cancel{z} \end{cases}$$

$$p = -\frac{1}{6}z^3 + q$$

$$0 = \begin{cases} +\frac{1}{6}\left(-\frac{1}{6^3}z^9\right) + \dots & \frac{9-9}{2} = 0 \\ +\frac{3}{4}z^3\left(\frac{1}{36}z^6\right) + \dots & \frac{9-9}{2} = 0 \\ +\frac{1}{2}z\left(\frac{1}{36}z^6 - \frac{1}{3}qz^3 + \dots\right) & \frac{9-7}{2} = 1 \\ +\frac{35}{112}z^6\left(-\frac{1}{6}z^3\right) + \dots & \frac{9-9}{2} = 0 \\ +\frac{15}{40}z^4\left(-\frac{1}{6}z^3 + q\right) & \frac{9-7}{2} = 1 \\ +\frac{1}{2}z^2\left(-\frac{1}{6}z^3 + q\right) & \frac{9-5}{2} = 2 \\ -\frac{1}{6}z^3 + q \\ +\frac{35}{1152}z^9 \\ +\frac{5}{112}z^7 \\ +\frac{3}{40}z^5 \\ +\frac{1}{6}z^3 \\ +z \\ -z \end{cases}$$

add like terms...

$$0 = \left\{ \begin{array}{l} \left(-\frac{1}{1296} + \frac{1}{48} - \frac{35}{672} + \frac{35}{1152}\right)z^9 \\ \left(+\frac{1}{72} - \frac{15}{240} + \frac{5}{112}\right)z^7 \\ \left(+\frac{15}{40} - \frac{1}{6}\right)qz^4 \\ \left(-\frac{1}{12} + \frac{3}{40}\right)z^5 \\ +\frac{1}{2}z^2q \\ \cancel{-\frac{1}{6}z^3} + q \\ \cancel{+\frac{1}{6}z^3} \end{array} \right.$$

$$0 = \left\{ \begin{array}{l} \left(-\frac{1}{1296} + \frac{1}{48} - \frac{35}{672} + \frac{35}{1152}\right)z^9 \\ \left(+\frac{1}{72} - \frac{15}{240} + \frac{5}{112}\right)z^7 \\ \left(+\frac{15}{40} - \frac{1}{6}\right)qz^4 \\ \left(-\frac{1}{12} + \frac{3}{40}\right)z^5 \\ +\frac{1}{2}z^2q \\ \cancel{-\frac{1}{6}z^3} + q \\ \cancel{+\frac{1}{6}z^3} \end{array} \right.$$

Linear in q !

$$\frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 = q\left(1 + \frac{1}{2}z^2 + \frac{5}{24}z^4\right)$$

$$\frac{\frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9}{1 + \frac{1}{2}z^2 + \frac{5}{24}z^4} = q$$

Long Division!

$$\begin{array}{r}
 \left(1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 \right) \left(\frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 \right) - \frac{1}{120}z^5 \\
 \hline
 \frac{1}{120}z^5 - \frac{1}{240}z^7 - \frac{5}{2880}z^9 \\
 \hline
 0 - \frac{1}{5040}z^7 - \frac{1}{10368}z^9
 \end{array}$$

The diagram shows a long division process. The dividend is $1 + \frac{1}{2}z^2 + \frac{5}{24}z^4$ and the divisor is $\frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9$. The quotient is $\frac{1}{120}z^5$. The remainder is $\frac{1}{120}z^5 - \frac{1}{240}z^7 - \frac{5}{2880}z^9$. A horizontal line is drawn under the remainder, and the next step shows the remainder as $0 - \frac{1}{5040}z^7 - \frac{1}{10368}z^9$. Blue arrows and boxes highlight the dividend, the divisor, the quotient, and the remainder. A blue 'X' is placed above the division line.

$$\begin{array}{r}
 \textcircled{1} + \frac{1}{2}z^2 + \frac{5}{24}z^4 \quad / \quad \frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 \quad / \quad \frac{1}{120}z^5 - \textcircled{\frac{1}{5040}z^7} \\
 - \frac{1}{120}z^5 - \frac{1}{240}z^7 - \frac{5}{2880}z^9 + \\
 \hline
 0 \quad \textcircled{-\frac{1}{5040}z^7} - \frac{1}{10368}z^9
 \end{array}$$

An arrow points from the circled 1 to the circled 0 in the result line. Another arrow points from the circled $-\frac{1}{5040}z^7$ to the circled $\frac{1}{5040}z^7$ in the second line.

$$\begin{array}{r}
 \left(1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 \right) \left(\frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 \right) - \frac{1}{120}z^5 - \frac{1}{5040}z^7 \\
 - \frac{1}{120}z^5 - \frac{1}{240}z^7 - \frac{5}{2880}z^9 + \\
 \hline
 0 - \frac{1}{5040}z^7 - \frac{1}{10368}z^9 \\
 + \frac{1}{5040}z^7 + \frac{1}{10080}z^9 \\
 \hline
 0 \quad \frac{1}{362880}z^9
 \end{array}$$

A blue 'X' is drawn above the first two terms of the first line. A blue arrow points from the 'X' to the first term of the first line. Another blue arrow points from the first term of the first line to the first term of the second line. A blue oval encircles the terms $-\frac{1}{5040}z^7 - \frac{1}{10368}z^9$ in the second line.

$$\begin{array}{r}
 \textcircled{1} + \frac{1}{2}z^2 + \frac{5}{24}z^4 \quad / \quad \frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 \quad \setminus \quad \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \textcircled{\frac{1}{362880}z^9} \\
 \hline
 - \frac{1}{120}z^5 - \frac{1}{240}z^7 - \frac{5}{2880}z^9 + \\
 \hline
 0 \quad - \frac{1}{5040}z^7 - \frac{1}{10368}z^9 \\
 \hline
 + \frac{1}{5040}z^7 + \frac{1}{10080}z^9 \\
 \hline
 0 \quad \textcircled{\frac{1}{362880}z^9}
 \end{array}$$

A blue arrow points from the circled '1' to the circled term $\frac{1}{362880}z^9$. A blue arrow points from the circled term $\frac{1}{362880}z^9$ to a '%' symbol.

Long division:

$$q = \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$$

$$x = z - \frac{1}{6}z^3 + q$$

$$x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 + O(z^{11})$$

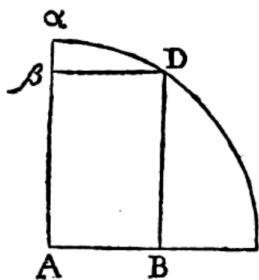
Long division:

$$q = \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$$

$$x = z - \frac{1}{6}z^3 + q$$

$$x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 + O(z^{11})$$

$x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$		
$x = z + p$	$ \begin{aligned} &+ \frac{35}{1152}x^9 \\ &+ \frac{5}{112}x^7 \\ &+ \frac{3}{40}x^5 \\ &+ \frac{1}{6}x^3 \\ &+ x \\ &- z \end{aligned} $	$ \begin{aligned} &+ \frac{35}{1152}z^9 + \dots \\ &+ \frac{5}{112}(z^7 + 7z^6p) + \dots \\ &+ \frac{3}{40}(z^5 + 5z^4p + 10z^3p^2) + \dots \\ &+ \frac{1}{6}(z^3 + 3z^2p + 3zp^2 + p^3) \\ &+ z + p \\ &- z \end{aligned} $
$p = -\frac{1}{6}z^3 + q$	$ \begin{aligned} &+ \frac{1}{6}p^3 \\ &+ \frac{3}{4}z^3p^2 \\ &+ \frac{1}{2}zp^2 \\ &+ \frac{35}{112}z^6p \\ &+ \frac{15}{40}z^4p \\ &+ \frac{1}{2}z^2p \\ &p \\ &+ \frac{35}{1152}z^9 \\ &+ \frac{5}{112}z^7 \\ &+ \frac{3}{40}z^5 \\ &+ \frac{1}{6}z^3 \\ &+z \\ &-z \end{aligned} $	$ \begin{aligned} &- \frac{1}{1296}z^9 + \dots \\ &+ \frac{1}{48}z^9 + \dots \\ &+ \frac{1}{72}z^7 - \frac{1}{6}qz^4 + \dots \\ &- \frac{35}{672}z^9 + \dots \\ &- \frac{15}{240}z^7 + \frac{15}{40}qz^4 + \dots \\ &- \frac{1}{12}z^5 + \frac{1}{2}z^2q \\ &- \frac{1}{6}z^3 + q \\ &+ \frac{35}{1152}z^9 \\ &+ \frac{5}{112}z^7 \\ &+ \frac{3}{40}z^5 \\ &+ \frac{1}{6}z^3 \\ &+ z \\ &- z \end{aligned} $
$1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 \quad \Bigg/ \quad \frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 \quad \Bigg/ \quad \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$		



45. If from the Arch αD given the Sine AB was required; I extract the Root of the Equation found above, *viz.* $z = x + \frac{1}{6}x^3 + \frac{1}{40}x^5 + \frac{1}{112}x^7$ (it being supposed that $AB = x$, $\alpha D = z$, and $A\alpha = 1$) by which I find $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{161280}z^9$ &c. (1711)

$$x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \quad (2026)$$

**Finding $\cos(z)$ By Applying the
Binomial Series To $\sqrt{1 - \sin^2(z)}$**

Contents

1. Finding a Power Series For $z = \sin^{-1}(x)$
2. Inverting $z = \sin^{-1}(x)$ to find $x = \sin(z)$
3. Finding $\cos(z)$ By Applying the Binomial Series To $\sqrt{1 - \sin^2(z)}$

Questions?

Newton, Isaac, and John Stewart. *Sir Isaac Newton's Two Treatises of the Quadrature of Curves, And Analysis by Equations of an Infinite Number of Terms Explained*. Printed by James Bettenham for J. Nourse, 1745. https://books.google.it/books?id=noQ_AAAAcAAJ.