Newton’s Power Series For Sine & Cosine (1669/1711)

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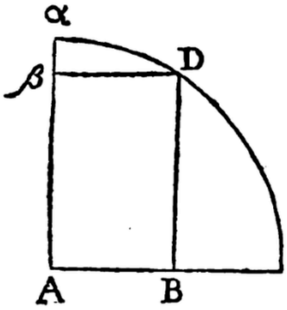

45. If from the Arch αD given the Sine AB was required; I extract the Root of the Equation found above, *viz.* $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$ (it being ſuppoſed that $AB = x$, $\alpha D = z$, and $A\alpha = 1$) by which I find $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$ &c. sin(z)

46. And moreover if the Coſine Aβ were required from that Arch given, make $A\beta$ $(= \sqrt{1 - xx}) = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6 + \frac{1}{40320}z^8 - \frac{1}{3628800}z^{10}$, &c. cos(z)

Newton, Isaac, and John Stewart. *Sir Isaac Newton's Two Treatises of the Quadrature of Curves, And Analysis by Equations of an Infinite Number of Terms Explained.* Printed by James Bettenham for J. Nourse, 1745. https://books.google.it/books?id=noQ_AAAAcAAJ (p. 338)

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Inverting $z = \sin^{-1}(x)$ to find $x = \sin(z)$

Inverting $z = \ln(1 + x)$

43. Thus if from the Area ABDC of the Hyperbola $\left(\frac{1}{1+x}=y\right)$ given I wanted to inveſtigate the Baſe AB, calling the Area z, I extract the Root of this Equation z (ABCD) $= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$, &c. neglecting thoſe Terms in which x is of more Dimenſions than z is deſired in the Quotient.

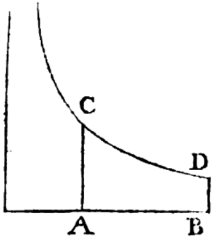
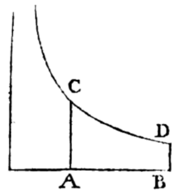


Figure 1: (Newton & Stewart, 1745, p. 337)

Goal:

$$x(z) = \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 \ldots$$



$$\ln(1+x) \equiv z(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \ldots$$

As if I would have z to riſe to five Dimenſions only in the Quotient, I neglect all the Terms $-\frac{1}{6}x^6+\frac{1}{7}x^7-\frac{1}{8}x^8$, &c. and extract the Root of this only $\frac{1}{5}x^5-\frac{1}{4}x^4+\frac{1}{3}x^3-\frac{1}{2}x^2+x-z=0$.

Figure 2: (Newton & Stewart, 1745, p. 337)

$$-z + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 = 0$$

$$0 = \begin{cases} +\frac{1}{5}x^5 \\ -\frac{1}{4}x^4 \\ +\frac{1}{3}x^3 \\ -\frac{1}{2}x^2 \\ + \quad x \\ - \quad z \end{cases}$$

$$x = z + p,$$

$$0 = \begin{cases} +\frac{1}{5}x^5 \\ -\frac{1}{4}x^4 \\ +\frac{1}{3}x^3 \\ -\frac{1}{2}x^2 \\ +\ x \\ -\ z \end{cases}$$

$$x = z + p,$$

$$0 = \begin{cases} +\frac{1}{5}(z+p)^5 \\ -\frac{1}{4}(z+p)^4 \\ +\frac{1}{3}(z+p)^3 \\ -\frac{1}{2}(z+p)^2 \\ + \quad z+p \\ - \quad z \end{cases}$$

$$
0 = \begin{cases}
+\frac{1}{5}z^{5} + z^{4}p + \dots & 5 - 5 = 0 \\
-\frac{1}{4}z^{4} \; \boxed{-z^{3}p} + \frac{3}{2}z^{2}p^{2} + \dots. & 5 - 4 = \boxed{1} \\
+\frac{1}{3}z^{3} \; \boxed{+z^{2}p + zp^{2}} + \frac{1}{3}zp^{3} + \dots. & 5 - 3 = \boxed{2} \\
-\frac{1}{2}z^{2} \; \boxed{-zp - \frac{1}{2}p^{2} + 0} & 5 - 2 = \boxed{3} \\
\quad \cancel{+z^{1}} + p & \\
\quad \cancel{-z^{1}} &
\end{cases}
$$

5,4,3 ...

5

"after the firſt Term reſulting from each Quantity that is collateral to it, I add no more Terms upon the right Hand than the Index of the Dimenſion of that firſt Term wants Units of the Index of the greateſt Dimenſion.

$$
0 = \begin{cases}
+\frac{1}{5}z^5 \\
-\frac{1}{4}z^4 - z^3p \\
+\frac{1}{3}z^3 + z^2p + zp^2 \\
-\frac{1}{2}z^2 - zp \quad - \frac{1}{2}p^2 \\
+z \qquad + p \\
-z
\end{cases}
$$

> “
>
> $$\begin{array}{llll}
> +\frac{1}{5}z^5 & \&c. & & \\
> -\frac{1}{4}z^4 & -z^3p & \&c. & \\
> +\frac{1}{3}z^3 & +z^2p & +zp^2 & \&c. \\
> -\frac{1}{2}z^2 & -zp & -\frac{1}{2}p^2 & \\
> +z & +p & & \\
> -z & & &
> \end{array}$$
>
> *Figure 3: (Newton & Stewart, 1745, p. 337)*

$$
\begin{array}{l}
+\ \tfrac{1}{5}z^5 \ \text{\&c.} \\
-\ \tfrac{1}{4}z^4 - z^3p \ \text{\&c.} \\
+\ \tfrac{1}{3}z^3 + z^2p + zp^2 \ \text{\&c.} \\
-\ \tfrac{1}{2}z^2 - zp - \tfrac{1}{2}p^2 \\
+\ z + p \\
-\ z
\end{array}
$$

$$p = \frac{1}{2}z^2 + O(z^3)$$

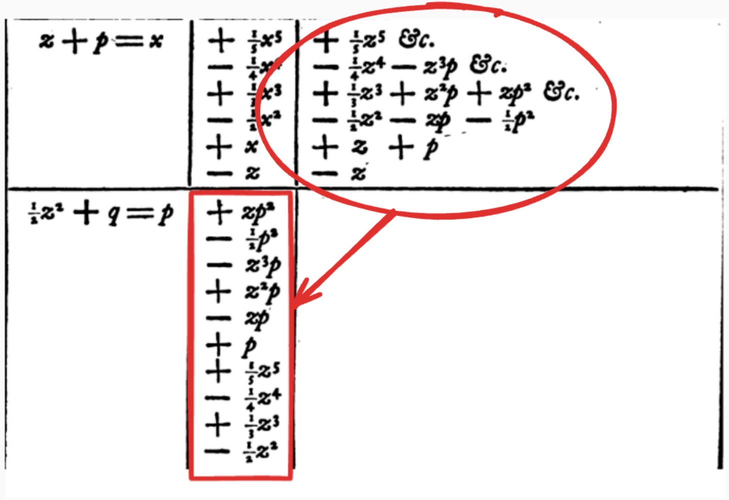
Figure 3: (Newton & Stewart, 1745, p. 337)

$z+p=x$	$+\frac{1}{5}x^5$	$+\frac{1}{5}z^5$ &c.
	$-\frac{1}{4}x^4$	$-\frac{1}{4}z^4-z^3p$ &c.
	$+\frac{1}{3}x^3$	$+\frac{1}{3}z^3+z^2p+zp^2$ &c.
	$-\frac{1}{2}x^2$	$-\frac{1}{2}z^2-zp-\frac{1}{2}p^2$
	$+x$	$+z\quad+p$
	$-z$	$-z$
$\frac{1}{2}z^2+q=p$		

$z + p = x$	$+ \frac{1}{5}x^5$	$+ \frac{1}{5}z^5$ &c.
	$- \frac{1}{4}x^4$	$- \frac{1}{4}z^4 - z^3p$ &c.
	$+ \frac{1}{3}x^3$	$+ \frac{1}{3}z^3 + z^2p + zp^2$ &c.
	$- \frac{1}{2}x^2$	$- \frac{1}{2}z^2 - zp - \frac{1}{2}p^2$
	$+ x$	$+ z + p$
	$- z$	$- z$
$\frac{1}{2}z^2 + q = p$		

$$0 = \left\{\begin{array}{l} +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 - z^3p \\ +\frac{1}{3}z^3 + z^2p + zp^2 \\ -\frac{1}{2}z^2 - zp \;\; - \frac{1}{2}p^2 \\ +z \qquad + p \\ -z \end{array}\right.$$

$$0 = \left\{\begin{array}{l} zp^2 \\ -\frac{1}{2}p^2 \\ -z^3p \\ +z^2p \\ -zp \\ +p \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{array}\right.$$



$z+p=x$	$+\frac{1}{5}x^5$ $-\frac{1}{4}x^4$ $+\frac{1}{3}x^3$ $-\frac{1}{2}x^2$ $+x$ $-z$	$+\frac{1}{5}z^5$ &c. $-\frac{1}{4}z^4-z^3p$ &c. $+\frac{1}{3}z^3+z^2p+zp^2$ &c. $-\frac{1}{2}z^2-zp-\frac{1}{2}p^2$ $+z\ +p$ $-z$
$\frac{1}{2}z^2+q=p$	$+zp^2$ $-\frac{1}{2}p^2$ $-z^3p$ $+z^2p$ $-zp$ $+p$ $+\frac{1}{5}z^5$ $-\frac{1}{4}z^4$ $+\frac{1}{3}z^3$ $-\frac{1}{2}z^2$	

Figure 4: (Newton & Stewart, 1745, p. 337)

$$
0 = \begin{cases}
+z\left(\frac{1}{2}z^2+q\right)^2 \\
-\frac{1}{2}\left(\frac{1}{2}z^2+q\right)^2 \\
-z^3\left(\frac{1}{2}z^2+q\right) \\
+z^2\left(\frac{1}{2}z^2+q\right) \\
-z\left(\frac{1}{2}z^2+q\right) \\
+\left(\frac{1}{2}z^2+q\right) \\
+\frac{1}{5}z^5 \\
-\frac{1}{4}z^4 \\
+\frac{1}{3}z^3 \\
-\frac{1}{2}z^2
\end{cases}
= \begin{cases}
\frac{1}{4}z^{5} + z^3 q & 5-5=0 \\
-\frac{1}{8}z^{4} \boxed{-\frac{1}{2}z^2 q} & 5-4=1 \\
-\frac{1}{2}z^{5} - z^3 q & 5-5=0 \\
+\frac{1}{2}z^{4} + \boxed{z^2 q} & 5-4=1 \\
-\frac{1}{2}z^{3} - qz & 5-3=2 \\
+\frac{1}{2}z^2 + q & \\
+\frac{1}{5}z^5 & \\
-\frac{1}{4}z^4 & \\
+\frac{1}{3}z^3 & \\
-\frac{1}{2}z^2 &
\end{cases}
$$

$$
0 = \begin{cases} +z\left(\frac{1}{2}z^2+q\right)^2 \\ -\frac{1}{2}\left(\frac{1}{2}z^2+q\right)^2 \\ -z^3\left(\frac{1}{2}z^2+q\right) \\ +z^2\left(\frac{1}{2}z^2+q\right) \\ -z\left(\frac{1}{2}z^2+q\right) \\ +\left(\frac{1}{2}z^2+q\right) \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases} = \begin{cases} \frac{1}{4}z^5 + z^3q \\ -\frac{1}{8}z^4 \boxed{-\frac{1}{2}z^2q} \\ -\frac{1}{2}z^5 - z^3q \\ +\frac{1}{2}z^4 + \boxed{z^2q} \\ -\frac{1}{2}z^3 - qz \\ +\frac{1}{2}z^2 + q \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases}
\qquad
\begin{array}{l} 5-5=0 \\ 5-4=1 \\ 5-5=0 \\ 5-4=1 \\ 5-3=2 \end{array}
$$

LINEAR in q

$$q = \frac{\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5}{1 - z + \frac{1}{2}z^2}$$

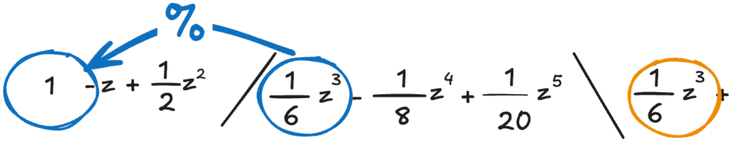
> 2. When I ſee that p, q, or r, *&c.* in the laſt reſulting Equation, is found of one Dimenſion only, I ſeek it's Value, that is to ſay the remaining Terms, which are ſtill to be added to the Quotient, by means of Diviſion; as you ſee done here.

Figure 5: (Newton & Stewart, 1745, p. 338)

> $$1 - z + \tfrac{1}{2}z^2)\ \tfrac{1}{6}z^3 - \tfrac{1}{8}z^4 + \tfrac{1}{20}z^5\ (\tfrac{1}{6}z^3 + \tfrac{1}{24}z^4 + \tfrac{1}{120}z^5$$

Figure 6: (Newton & Stewart, 1745, p. 337)

$$q = \frac{\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5}{1 - z + \frac{1}{2}z^2}$$



$$1 - z + \frac{1}{2}z^2 \;\Big)\; \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5 \;\Big(\; \frac{1}{6}z^3$$

$$-\frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5$$

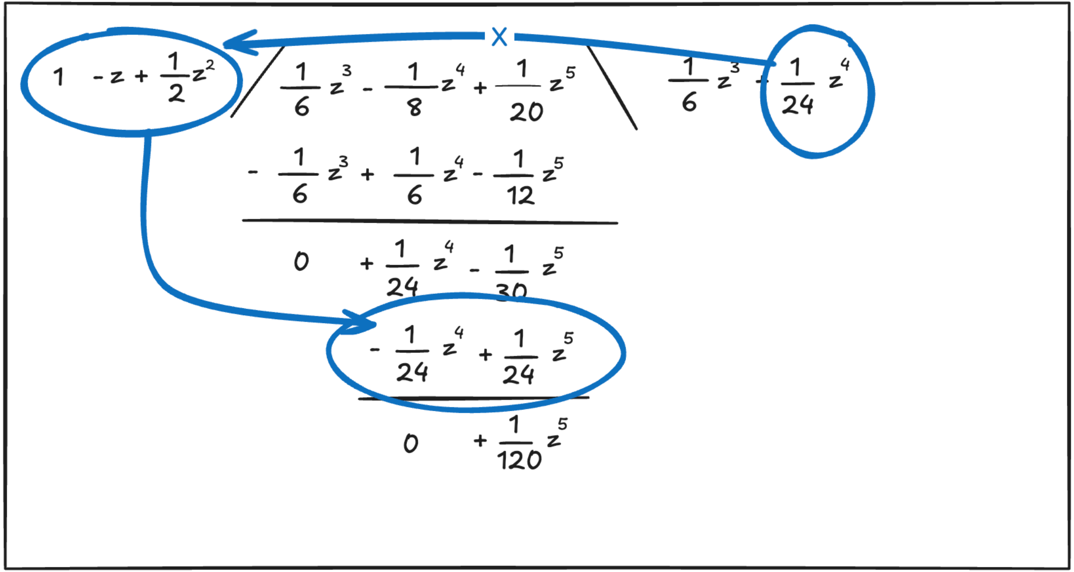
$$0 + \frac{1}{24}z^4 + \frac{1}{30}z^5$$

$$1 - z + \frac{1}{2}z^2 \;\Big/\; \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5 \;\Big\backslash\; \frac{1}{6}z^3 + \frac{1}{24}z^4$$

$$-\frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5$$

$$0 \quad + \frac{1}{24}z^4 - \frac{1}{30}z^5$$

%


$1 - z + \frac{1}{2}z^2$
$\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5$
$\frac{1}{6}z^3 + \frac{1}{24}z^4$
x
$-\frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5$
$0 + \frac{1}{24}z^4 - \frac{1}{30}z^5$
$-\frac{1}{24}z^4 + \frac{1}{24}z^5$
$0 + \frac{1}{120}z^5$

$$1 - z + \frac{1}{2}z^2 \Big/ \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5 \Big\backslash \quad \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$$

$$-\frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5$$

$$0 \quad + \frac{1}{24}z^4 - \frac{1}{30}z^5$$

$$-\frac{1}{24}z^4 + \frac{1}{24}z^5$$

$$0 \quad + \frac{1}{120}z^5$$

%

$$q = \frac{\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5}{1 - z + \frac{1}{2}z^2}$$

Long Division

$$q = \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 + O(z^7)$$

Long division:

$$q = \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 + O(z^7)$$

$$x = z + \frac{1}{2}z^2 + q$$

$$e^z - 1 \equiv x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 + \ldots$$

$x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$ &c.		
$z + p = x$	$+ \frac{1}{5}x^5$	$+ \frac{1}{5}z^5$ &c.
	$- \frac{1}{4}x^4$	$- \frac{1}{4}z^4 - z^3p$ &c.
	$+ \frac{1}{3}x^3$	$+ \frac{1}{3}z^3 + z^2p + zp^2$ &c.
	$- \frac{1}{2}x^2$	$- \frac{1}{2}z^2 - zp - \frac{1}{2}p^2$
	$+ x$	$+ z + p$
	$- z$	$- z$
$\frac{1}{2}z^2 + q = p$	$+ zp^2$	$+ \frac{1}{4}z^5$ &c.
	$- \frac{1}{2}p^2$	$- \frac{1}{8}z^4 - \frac{1}{2}z^2q$
	$- z^3p$	$- \frac{1}{2}z^5$ &c.
	$+ z^2p$	$+ \frac{1}{2}z^4 + z^2q$
	$- zp$	$- \frac{1}{2}z^3 - zq$
	$+ p$	$+ \frac{1}{2}z^2 + q$
	$+ \frac{1}{5}z^5$	$+ \frac{1}{5}z^5$
	$- \frac{1}{4}z^4$	$- \frac{1}{4}z^4$
	$+ \frac{1}{3}z^3$	$+ \frac{1}{3}z^3$
	$- \frac{1}{2}z^2$	$- \frac{1}{2}z^2$
$1 - z + \frac{1}{2}z^2)\ \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5\ (\frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$		

Figure 19: (Newton & Stewart, 1745, p. 337)

Inverting $z = \sin^{-1}(x)$

infinite

$$\sin^{-1}(x) \equiv z(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \ldots$$

finite

$$z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9$$

$$z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9$$

rearrange:

$$0 = \begin{cases} +\frac{35}{1152}x^9 \\ +\frac{5}{112}x^7 \\ +\frac{3}{40}x^5 \\ +\frac{1}{6}x^3 \\ +x \\ -z \end{cases}$$

$$x = z + p$$

substitute

$$x = z + p$$

$$0 = \begin{cases} +\frac{35}{1152}(z+p)^9 \\ +\frac{5}{112}(z+p)^7 \\ +\frac{3}{40}(z+p)^5 \\ +\frac{1}{6}(z+p)^3 \\ +x \\ -z \end{cases}$$

$$
0 = \begin{cases}
+\frac{35}{1152}\left(z^{9} + 9z^{8}p\right) + \ldots & \frac{9-9}{2} = 0 \\
+\frac{5}{112}\left(z^{7} + 7z^{6}p + 28z^{5}p^{2}\right) + \ldots & \frac{9-7}{2} = 1 \\
+\frac{3}{40}\left(z^{5} + 5z^{4}p + 10z^{3}p^{2} + 10z^{2}p^{3}\right) + \ldots & \frac{9-5}{2} = 2 \\
+\frac{1}{6}\left(z^{3} + 3z^{2}p + 3zp^{2} + p^{3}\right) + \ldots & \frac{9-3}{2} = 3 \\
+z + p & \\
-z &
\end{cases}
$$

9,7,5,3... 9 % 2

after the firſt Term, reſulting from each Quantity which is collateral to it, you add no more Terms towards the Right Hand, than what the Index of the Dimenſion of that firſt Term, wants Pairs of Units of the Index of the higheſt Dimenſion;

$$
0 = \begin{cases}
+\frac{35}{1152} z^9 \\
+\frac{5}{112}(z^7 + 7z^6 p) \\
+\frac{3}{40}(z^5 \quad + 5z^4 p + 10z^3 p^2) \\
+\frac{1}{6}(z^3 \quad + 3z^2 p + 3zp^2 + p^3) \\
+z \quad + p \\
-z
\end{cases}
$$

$$p = -\frac{1}{6}z^3 + O(z^5)$$

$$0 = \begin{cases} +\frac{35}{1152}z^9 \\ +\frac{5}{112}(z^7 + 7z^6 p) \\ +\frac{3}{40}(z^5 + 5z^4 p + 10z^3 p^2) \\ +\frac{1}{6}(z^3 + 3z^2 p + 3zp^2 + p^3) \\ +z + p \\ -z \end{cases}$$

$$
0 = \begin{cases}
+\frac{35}{1152}z^9 \\
+\frac{5}{112}(z^7 + 7z^6p) \\
+\frac{3}{40}(z^5 + 5z^4p + 10z^3p^2) \\
+\frac{1}{6}(z^3 + 3z^2p + 3zp^2 + p^3) \\
+z \quad + p \\
-z
\end{cases}
$$

rearrange

$$
0 = \begin{cases}
+\frac{1}{6}p^3 \\
+\frac{3}{4}z^3p^2 \\
+\frac{1}{2}zp^2 \\
+\frac{35}{112}z^6p \\
+\frac{15}{40}z^4p \\
+\frac{1}{2}z^2p \\
p \\
+\frac{35}{1152}z^9 \\
+\frac{5}{112}z^7 \\
+\frac{3}{40}z^5 \\
+\frac{1}{6}z^3 \\
\cancel{+z} \\
\cancel{-z}
\end{cases}
$$

$$0 = \begin{cases} +\frac{1}{6}p^3 \\ +\frac{3}{4}z^3p^2 \\ +\frac{1}{2}zp^2 \\ +\frac{35}{112}z^6p \\ +\frac{15}{40}z^4p \\ +\frac{1}{2}z^2p \\ p \\ +\frac{35}{1152}z^9 \\ +\frac{5}{112}z^7 \\ +\frac{3}{40}z^5 \\ +\frac{1}{6}z^3 \\ ~~+z~~ \\ ~~-z~~ \end{cases}$$

substitute

$$p = -\frac{1}{6}z^3 + q$$

$$0 = \begin{cases} +\frac{1}{6}\left(-\frac{1}{6^3}z^9\right) + \ldots & \frac{9-9}{2} = 0 \\ +\frac{3}{4}z^3\left(\frac{1}{36}z^6\right) + \ldots & \frac{9-9}{2} = 0 \\ +\frac{1}{2}z\left(\frac{1}{36}z^6 - \frac{1}{3}qz^3 + \ldots\right) & \frac{9-7}{2} = 1 \\ +\frac{35}{112}z^6\left(-\frac{1}{6}z^3\right) + \ldots & \frac{9-9}{2} = 0 \\ +\frac{15}{40}z^4\left(-\frac{1}{6}z^3 + q\right) & \frac{9-7}{2} = 1 \\ +\frac{1}{2}z^2\left(-\frac{1}{6}z^3 + q\right) & \frac{9-5}{2} = 2 \\ -\frac{1}{6}z^3 + q & \\ +\frac{35}{1152}z^9 & \\ +\frac{5}{112}z^7 & \\ +\frac{3}{40}z^5 & \\ +\frac{1}{6}z^3 & \\ +z & \\ -z & \end{cases}$$

add like terms...

$$0 = \begin{cases} \left(-\frac{1}{1296} + \frac{1}{48} - \frac{35}{672} + \frac{35}{1152}\right) z^9 \\ \left(+\frac{1}{72} - \frac{15}{240} + \frac{5}{112}\right) z^7 \\ \left(+\frac{15}{40} - \frac{1}{6}\right) q z^4 \\ \left(-\frac{1}{12} + \frac{3}{40}\right) z^5 \\ +\frac{1}{2} z^2 q \\ \cancel{-\frac{1}{6} z^3} + q \\ \cancel{+\frac{1}{6} z^3} \end{cases}$$

$$0 = \begin{cases} \left(-\frac{1}{1296} + \frac{1}{48} - \frac{35}{672} + \frac{35}{1152}\right) z^9 \\ \left(+\frac{1}{72} - \frac{15}{240} + \frac{5}{112}\right) z^7 \\ \left(+\frac{15}{40} - \frac{1}{6}\right) q z^4 \\ \left(-\frac{1}{12} + \frac{3}{40}\right) z^5 \\ +\frac{1}{2} z^2 q \\ \cancel{-\frac{1}{6} z^3} + q \\ \cancel{+\frac{1}{6} z^3} \end{cases}$$

Linear in q!

$$\frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 = q\left(1 + \frac{1}{2}z^2 + \frac{5}{24}z^4\right)$$

$$\frac{\frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9}{1 + \frac{1}{2}z^2 + \frac{5}{24}z^4} = q$$

Long Division!

$$1 + \frac{1}{2}z^{2} + \frac{5}{24}z^{4} \;/\; \frac{1}{120}z^{5} + \frac{1}{252}z^{7} + \frac{17}{10368}z^{9} \;\backslash\; \frac{1}{120}z^{5} +$$

%

$$1+\frac{1}{2}z^2+\frac{5}{24}z^4 \;\Big)\; \frac{1}{120}z^5+\frac{1}{252}z^7+\frac{17}{10368}z^9 \;\Big(\; \frac{1}{120}z^5$$

×

$$\frac{1}{120}z^5-\frac{1}{240}z^7-\frac{5}{2880}z^9$$

$$0\;-\frac{1}{5040}z^7-\frac{1}{10368}z^9$$

$$1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 \Big/ \frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 \Big\backslash \frac{1}{120}z^5 - \frac{1}{5040}z^7$$

$$-\frac{1}{120}z^5 - \frac{1}{240}z^7 - \frac{5}{2880}z^9 +$$

$$0 \quad -\frac{1}{5040}z^7 - \frac{1}{10368}z^9$$

%

$$1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 \;\Big/\; \frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 \;\Big\backslash\; \frac{1}{120}z^5 - \frac{1}{5040}z^7$$

X

$$-\frac{1}{120}z^5 - \frac{1}{240}z^7 - \frac{5}{2880}z^9 +$$

$$0 \quad -\frac{1}{5040}z^7 - \frac{1}{10368}z^9$$

$$+\frac{1}{5040}z^7 + \frac{1}{10080}z^9$$

$$0 \quad \frac{1}{362880}z^9$$

$$1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 \;\Big/\; \frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 \;\Big\backslash\; \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$$

$$-\;\frac{1}{120}z^5 - \frac{1}{240}z^7 - \frac{5}{2880}z^9 \;+$$

$$0 \quad -\frac{1}{5040}z^7 - \frac{1}{10368}z^9$$

$$+\frac{1}{5040}z^7 + \frac{1}{10080}z^9$$

$$0 \quad \frac{1}{362880}z^9$$

%

Long division:

$$q = \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$$

$$x = z - \frac{1}{6}z^3 + q$$

$$x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 + O(z^{11})$$

Long division:

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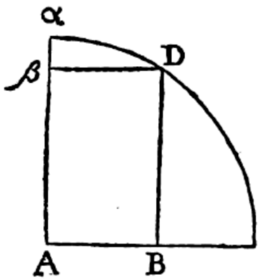
$$x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$$

$x = z + p$	$+\frac{35}{1152}x^9$ $+\frac{5}{112}x^7$ $+\frac{3}{40}x^5$ $+\frac{1}{6}x^3$ $+x$ $-z$	$+\frac{35}{1152}z^9 + \ldots$ $+\frac{5}{112}(z^7 \quad + 7z^6p) + \ldots$ $+\frac{3}{40}(\ z^5 \quad + 5z^4p + 10z^3p^2) + \ldots$ $+\frac{1}{6}(\ z^3 \quad + 3z^2p + 3zp^2 + p^3)$ $+z + p$ $-z$
$p = -\frac{1}{6}z^3 + q$	$+\frac{1}{6}p^3$ $+\frac{3}{4}z^3p^2$ $+\frac{1}{2}zp^2$ $+\frac{35}{112}z^6p$ $+\frac{15}{40}z^4p$ $+\frac{1}{2}z^2p$ p $+\frac{35}{1152}z^9$ $+\frac{5}{112}z^7$ $+\frac{3}{40}z^5$ $+\frac{1}{6}z^3$ ~~$+z$~~ ~~$-z$~~	$-\frac{1}{1296}z^9 + \ldots$ $+\frac{1}{48}z^9 + \ldots$ $+\frac{1}{72}z^7 \quad - \frac{1}{6}qz^4 + \ldots$ $-\frac{35}{672}z^9 + \ldots$ $-\frac{15}{240}z^7 \quad + \frac{15}{40}qz^4 + \ldots$ $-\frac{1}{12}z^5 \quad + \frac{1}{2}z^2q$ $-\frac{1}{6}z^3 + q$ $+\frac{35}{1152}z^9$ $+\frac{5}{112}z^7$ $+\frac{3}{40}z^5$ $+\frac{1}{6}z^3$ $+z$ $-z$

$$1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 \Big/ \frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 \Big\backslash \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$$



45. If from the Arch αD given the Sine AB was required; I extract the Root of the Equation found above, *viz.* $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$ (it being supposed that $AB = x$, $\alpha D = z$, and $A\alpha = 1$) by which I find $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$ &c. (1711)

$$x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

(2026)

Finding $\cos(z)$ By Applying the Binomial Series To $\sqrt{1 - \sin^2(z)}$

Contents

Questions?

Newton, Isaac, and John Stewart. *Sir Isaac Newton's Two Treatises of the Quadrature of Curves, And Analysis by Equations of an Infinite Number of Terms Explained.* Printed by James Bettenham for J. Nourse, 1745. https://books.google.it/books?id=noQ_AAAAcAAJ.