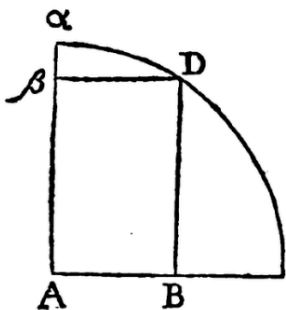


Newton's Power Series For Sine & Cosine (1669/1711)

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45. If from the Arch αD given the Sine AB was required; I extract the Root of the Equation found above, *viz.* $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$ (it being supposed that $AB = x$, $\alpha D = z$, and $A\alpha = 1$) by which I find $x = z - \frac{1}{6}z^3 +$

$$\frac{1}{120}z^5 - \frac{1}{5640}z^7 + \frac{1}{362880}z^9 \text{ \&c.}$$

$\sin(z)$

46. And moreover if the Cosine $A\beta$ were required from that Arch given, make $A\beta (=$

$$\sqrt{1 - xx}) = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6 + \frac{1}{40320}z^8 - \frac{1}{3628800}z^{10}, \text{ \&c.}$$

$\cos(z)$

Contents

1. Finding a Power Series For $z = \sin^{-1}(x)$
2. Inverting $z = \sin^{-1}(x)$ to find $x = \sin(z)$
3. Finding $\cos(z)$ By Applying the Binomial Series To $\sqrt{1 - \sin^2(z)}$

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Finding a Power Series For $z = \sin^{-1}(x)$

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Inverting $z = \sin^{-1}(x)$ to find $x = \sin(z)$

Inverting $z = \ln(1 + x)$

43. Thus if from the Area ABDC of the Hyperbola ($\frac{1}{1+x} = y$) given I wanted to investigate the Base AB, calling the Area z , I extract the Root of this Equation $z(ABCD) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4, \&c.$ neglecting those Terms in which x is of more Dimensions than z is desired in the Quotient.

As if I would have z to rise to five Dimen-

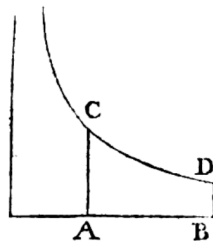
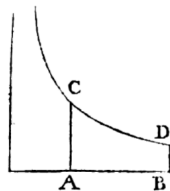


Figure 1: (Newton & Stewart, 1745, p. 337)

Goal:

$$x(z) = \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 \dots$$



$$\ln(1+x) \equiv z(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \dots$$

As if \sqrt{x} would have z to rise to five Dimensions only in the Quotient, I neglect all the Terms $-\frac{1}{8}x^6 + \frac{1}{7}x^7 - \frac{1}{4}x^8$, &c. and extract the Root of this only $\frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - z = 0$.

Figure 2: (Newton & Stewart, 1745, p. 337)

$$-z + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 = 0$$

$$0 = \begin{cases} +\frac{1}{5}x^5 \\ -\frac{1}{4}x^4 \\ +\frac{1}{3}x^3 \\ -\frac{1}{2}x^2 \\ + x \\ - z \end{cases}$$

$$x = z + p,$$

$$0 = \begin{cases} +\frac{1}{5}x^5 \\ -\frac{1}{4}x^4 \\ +\frac{1}{3}x^3 \\ -\frac{1}{2}x^2 \\ + x \\ - z \end{cases}$$

$$x = z + p,$$

$$0 = \begin{cases} +\frac{1}{5}(z+p)^5 \\ -\frac{1}{4}(z+p)^4 \\ +\frac{1}{3}(z+p)^3 \\ -\frac{1}{2}(z+p)^2 \\ + z + p \\ - z \end{cases}$$

$$0 = \left\{ \begin{array}{l}
+\frac{1}{5}z^5 + z^4p + \dots \\
-\frac{1}{4}z^4 \boxed{-z^3p} + \frac{3}{2}z^2p^2 + \dots \\
+\frac{1}{3}z^3 \boxed{+z^2p + zp^2} + \frac{1}{3}zp^3 + \dots \\
-\frac{1}{2}z^2 \boxed{-zp - \frac{1}{2}p^2 + 0} \\
\cancel{+z^1} + p \\
\cancel{-z^1}
\end{array} \right. \begin{array}{l}
5 - 5 = 0 \\
5 - 4 = \boxed{1} \\
5 - 3 = \boxed{2} \\
5 - 2 = \boxed{3}
\end{array}$$

5,4,3 ...

5

after the first Term resulting from each Quantity that is collateral to it, I add no more Terms upon the right Hand than the Index of the Dimension of that first Term wants Units of the Index of the greatest Dimension.

$$0 = \begin{cases} +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 - z^3p \\ +\frac{1}{3}z^3 + z^2p + zp^2 \\ -\frac{1}{2}z^2 - zp - \frac{1}{2}p^2 \\ +z + p \\ -z \end{cases}$$

$$\begin{array}{r}
+ \frac{1}{5}z^5 \quad \text{etc.} \\
- \frac{1}{4}z^4 - z^3p \quad \text{etc.} \\
+ \frac{1}{3}z^3 + z^2p + zp^2 \quad \text{etc.} \\
- \frac{1}{2}z^2 - zp - \frac{1}{2}p^2 \\
+ z + p \\
- z
\end{array}$$

Figure 3: (Newton & Stewart, 1745, p. 337)

$$p = \frac{1}{2}z^2 + O(z^3)$$

$$+ \frac{1}{5}z^5 \text{ etc.}$$

$$- \frac{1}{4}z^4 - z^3p \text{ etc.}$$

$$+ \frac{1}{3}z^3 + z^2p + zp^2 \text{ etc.}$$

$$- \frac{1}{2}z^2 - zp - \frac{1}{2}p^2$$

$$+ z + p$$

$$- z$$

Figure 3: (Newton & Stewart, 1745, p. 337)

$$x + p = x$$

$$+ \frac{1}{5}x^5$$

$$- \frac{1}{4}x^4$$

$$+ \frac{1}{3}x^3$$

$$- \frac{1}{2}x^2$$

$$+ x$$

$$- x$$

$$+ \frac{1}{5}z^5 \text{ } \mathcal{E}c.$$

$$- \frac{1}{4}z^4 - z^3p \text{ } \mathcal{E}c.$$

$$+ \frac{1}{3}z^3 + z^2p + zp^2 \text{ } \mathcal{E}c.$$

$$- \frac{1}{2}z^2 - zp - \frac{1}{2}p^2$$

$$+ z + p$$

$$- z$$

$$\frac{1}{2}z^2 + q = p$$

$$x + p = x$$

$$+ \frac{1}{5}x^5$$

$$- \frac{1}{4}x^4$$

$$+ \frac{1}{3}x^3$$

$$- \frac{1}{2}x^2$$

$$+ x$$

$$- x$$

$$+ \frac{1}{5}x^5 \text{ \Ô.}$$

$$- \frac{1}{4}x^4 - x^3p \text{ \Ô.}$$

$$+ \frac{1}{3}x^3 + x^2p + xp^2 \text{ \Ô.}$$

$$- \frac{1}{2}x^2 - xp - \frac{1}{2}p^2$$

$$+ x + p$$

$$- x$$

$$\frac{1}{2}x^2 + q = p$$



$$0 = \begin{cases} +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 - z^3p \\ +\frac{1}{3}z^3 + z^2p + zp^2 \\ -\frac{1}{2}z^2 - zp - \frac{1}{2}p^2 \\ +z + p \\ -z \end{cases}$$



$$0 = \begin{cases} zp^2 \\ -\frac{1}{2}p^2 \\ -z^3p \\ +z^2p \\ -zp \\ +p \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases}$$

$x + p = x$	$+$ $\frac{1}{5}x^5$ $-$ $\frac{1}{4}x^4$ $+$ $\frac{1}{3}x^3$ $-$ $\frac{1}{2}x^2$ $+$ x $-$ x	$+$ $\frac{1}{5}x^5$ C.c. $-$ $\frac{1}{4}x^4 - x^3p$ C.c. $+$ $\frac{1}{3}x^3 + x^2p + xp^2$ C.c. $-$ $\frac{1}{2}x^2 - xp - \frac{1}{2}p^2$ $+$ $x + p$ $-$ x
$\frac{1}{2}x^2 + q = p$	$+$ xp^2 $-$ $\frac{1}{2}p^2$ $-$ x^3p $+$ x^2p $-$ xp $+$ p $+$ $\frac{1}{5}x^5$ $-$ $\frac{1}{4}x^4$ $+$ $\frac{1}{3}x^3$ $-$ $\frac{1}{2}x^2$	

Figure 4: (Newton & Stewart, 1745, p. 337)

$$0 = \begin{cases} +z\left(\frac{1}{2}z^2 + q\right)^2 \\ -\frac{1}{2}\left(\frac{1}{2}z^2 + q\right)^2 \\ -z^3\left(\frac{1}{2}z^2 + q\right) \\ +z^2\left(\frac{1}{2}z^2 + q\right) \\ -z\left(\frac{1}{2}z^2 + q\right) \\ +\left(\frac{1}{2}z^2 + q\right) \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases} = \begin{cases} \frac{1}{4}z^5 + z^3q \\ -\frac{1}{8}z^4 - \frac{1}{2}z^2q \\ -\frac{1}{2}z^5 - z^3q \\ +\frac{1}{2}z^4 + z^2q \\ -\frac{1}{2}z^3 - qz \\ +\frac{1}{2}z^2 + q \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases} \begin{matrix} 5 - 5 = 0 \\ 5 - 4 = 1 \\ 5 - 5 = 0 \\ 5 - 4 = 1 \\ 5 - 3 = 2 \end{matrix}$$

$$0 = \begin{cases} +z\left(\frac{1}{2}z^2 + q\right)^2 \\ -\frac{1}{2}\left(\frac{1}{2}z^2 + q\right)^2 \\ -z^3\left(\frac{1}{2}z^2 + q\right) \\ +z^2\left(\frac{1}{2}z^2 + q\right) \\ -z\left(\frac{1}{2}z^2 + q\right) \\ +\left(\frac{1}{2}z^2 + q\right) \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases} = \begin{cases} \frac{1}{4}z^5 + z^3q & 5 - 5 = 0 \\ -\frac{1}{8}z^4 - \frac{1}{2}z^2q & 5 - 4 = 1 \\ -\frac{1}{2}z^5 - z^3q & 5 - 5 = 0 \\ +\frac{1}{2}z^4 + z^2q & 5 - 4 = 1 \\ -\frac{1}{2}z^3 - qz & 5 - 3 = 2 \\ +\frac{1}{2}z^2 + q \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases}$$

LINEAR in q

$$q = \frac{\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5}{1 - z + \frac{1}{2}z^2}$$

2. When I see that p , q , or r , &c. in the last resulting Equation, is found of one Dimension only, I seek it's Value, that is to say the remaining Terms, which are still to be added to the Quotient, by means of Division; as you see done here.

Figure 5: (Newton & Stewart, 1745, p. 338)

$$1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 \quad \left(\frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 \right)$$

Figure 6: (Newton & Stewart, 1745, p. 337)

$$q = \frac{\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5}{1 - z + \frac{1}{2}z^2}$$

Diagram illustrating the long division process for the rational function q . The denominator is $1 - z + \frac{1}{2}z^2$. The numerator is $\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5$. The diagram shows the first step of division, where the leading term of the numerator, $\frac{1}{6}z^3$, is divided by the leading term of the denominator, 1 . The result of this division is $\frac{1}{6}z^3$, which is circled in orange. The terms 1 and $\frac{1}{6}z^3$ are circled in blue. A blue arrow points from the circled 1 to the denominator, and another blue arrow points from the circled $\frac{1}{6}z^3$ to the denominator. A blue '%' symbol is positioned above the division line.

$$1 - z + \frac{1}{2}z^2 \overline{) \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5}$$

$$\begin{array}{r}
 \left(1 - z + \frac{1}{2}z^2 \right) \left(\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5 \right) \\
 \left(-\frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5 \right) \\
 \hline
 0 + \frac{1}{24}z^4 + \frac{1}{30}z^5
 \end{array}$$

The diagram shows a multiplication of two polynomials. The first polynomial is $1 - z + \frac{1}{2}z^2$ and the second is $\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5$. The result of the multiplication is $-\frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5$. A horizontal line is drawn below the result, and the final simplified result is $0 + \frac{1}{24}z^4 + \frac{1}{30}z^5$. Blue circles and arrows highlight the terms $1 - z + \frac{1}{2}z^2$ and $\frac{1}{6}z^3$ in the first row, and $-\frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5$ in the second row. A blue 'X' is placed above the multiplication symbol, and blue arrows point from the 'X' to the circled terms.

$$\begin{array}{r}
 \textcircled{1} - z + \frac{1}{2}z^2 \quad / \quad \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5 \quad / \quad \frac{1}{6}z^3 + \textcircled{\frac{1}{24}z^4} \\
 - \frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5 \\
 \hline
 0 \quad \textcircled{+\frac{1}{24}z^4} - \frac{1}{30}z^5
 \end{array}$$

$$1 - z + \frac{1}{2}z^2$$

$$\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5$$

$$\frac{1}{6}z^3 - \frac{1}{24}z^4$$

$$- \frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5$$

$$0 + \frac{1}{24}z^4 - \frac{1}{30}z^5$$

$$- \frac{1}{24}z^4 + \frac{1}{24}z^5$$

$$0 + \frac{1}{120}z^5$$

X



$$1 - z + \frac{1}{2}z^2$$

$$\left/ \begin{array}{l} \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5 \\ - \frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5 \end{array} \right.$$

$$\frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$$

$$\hline - \frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5$$

$$\hline 0 + \frac{1}{24}z^4 - \frac{1}{30}z^5$$

$$- \frac{1}{24}z^4 + \frac{1}{24}z^5$$

$$\hline 0 + \frac{1}{120}z^5$$

%



$$q = \frac{\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5}{1 - z + \frac{1}{2}z^2}$$

Long Division

$$q = \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 + O(z^7)$$

Long division:

$$q = \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 + O(z^7)$$

$$x = z + \frac{1}{2}z^2 + q$$

$$e^z - 1 \equiv x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 + \dots$$

$x = x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 \text{ \&c.}$		
$x + p = x$	$+$ $\frac{1}{2}x^2$ $-$ $\frac{1}{4}x^4$ $+$ $\frac{1}{12}x^3$ $-$ $\frac{1}{8}x^2$ $+$ x $-$ x	$+$ $\frac{1}{2}x^2 \text{ \&c.}$ $-$ $\frac{1}{4}x^4 - x^2p \text{ \&c.}$ $+$ $\frac{1}{12}x^3 + x^2p + xp^2 \text{ \&c.}$ $-$ $\frac{1}{8}x^2 - xp - \frac{1}{2}p^2$ $+$ $x + p$ $-$ x
$\frac{1}{2}x^2 + q = p$	$+$ xp^2 $-$ $\frac{1}{2}p^3$ $-$ x^3p $+$ x^2p $-$ xp $+$ p $+$ $\frac{1}{2}x^5$ $-$ $\frac{1}{4}x^4$ $+$ $\frac{1}{12}x^3$ $-$ $\frac{1}{8}x^2$	$+$ $\frac{1}{2}x^5 \text{ \&c.}$ $-$ $\frac{1}{4}x^4 - \frac{1}{2}x^2q$ $-$ $\frac{1}{12}x^5 \text{ \&c.}$ $+$ $\frac{1}{2}x^4 + x^2q$ $-$ $\frac{1}{8}x^3 - xq$ $+$ $\frac{1}{2}x^2 + q$ $+$ $\frac{1}{2}x^5$ $-$ $\frac{1}{4}x^4$ $+$ $\frac{1}{12}x^3$ $-$ $\frac{1}{8}x^2$
$1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 (\frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$		

Figure 19: (Newton & Stewart, 1745, p. 337)

Inverting $z = \sin^{-1}(x)$.

infinite

$$\sin^{-1}(x) \equiv z(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \dots$$

finite

$$z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9$$

$$z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9$$

rearrange:

$$x = z + p$$

$$0 = \left\{ \begin{array}{l} + \frac{35}{1152}x^9 \\ + \frac{5}{112}x^7 \\ + \frac{3}{40}x^5 \\ + \frac{1}{6}x^3 \\ + x \\ - z \end{array} \right.$$

substitute

$$x = z + p$$

$$0 = \begin{cases} +\frac{35}{1152}(z+p)^9 \\ +\frac{5}{112}(z+p)^7 \\ +\frac{3}{40}(z+p)^5 \\ +\frac{1}{6}(z+p)^3 \\ +x \\ -z \end{cases}$$

$$0 = \begin{cases} +\frac{35}{1152}(z^9 + 9z^8p) + \dots & \frac{9-9}{2} = 0 \\ +\frac{5}{112}(z^7 + 7z^6p + 28z^5p^2) + \dots & \frac{9-7}{2} = 1 \\ +\frac{3}{40}(z^5 + 5z^4p + 10z^3p^2 + 10z^2p^3) + \dots & \frac{9-5}{2} = 2 \\ +\frac{1}{6}(z^3 + 3z^2p + 3zp^2 + p^3) + \dots & \frac{9-3}{2} = 3 \\ +z + p \\ -z \end{cases}$$

9, 7, 5, 3...

9

9% 2

after the first Term, resulting from each Quantity which is collateral to it, you add no more Terms towards the Right Hand, than what the Index of the Dimension of that first Term, wants Pairs of Units of the Index of the highest Dimension;

$$0 = \left\{ \begin{array}{l} + \frac{35}{1152} z^9 \\ + \frac{5}{112} (z^7 + 7z^6 p) \\ + \frac{3}{40} (z^5 + 5z^4 p + 10z^3 p^2) \\ + \frac{1}{6} (z^3 + 3z^2 p + 3z p^2 + p^3) \\ + z \quad + p \\ - z \end{array} \right.$$

$$p = -\frac{1}{6}z^3 + O(z^5)$$

$$0 = \begin{cases} +\frac{35}{1152}z^9 \\ +\frac{5}{112}(z^7 + 7z^6p) \\ +\frac{3}{40}(z^5 + 5z^4p + 10z^3p^2) \\ +\frac{1}{6}(z^3 + 3z^2p + 3zp^2 + p^3) \\ +z + p \\ -z \end{cases}$$

rearrange

$$0 = \begin{cases} +\frac{35}{1152}z^9 \\ +\frac{5}{112}(z^7 + 7z^6p) \\ +\frac{3}{40}(z^5 + 5z^4p + 10z^3p^2) \\ +\frac{1}{6}(z^3 + 3z^2p + 3zp^2 + p^3) \\ +z + p \\ -z \end{cases}$$



$$0 = \begin{cases} +\frac{1}{6}p^3 \\ +\frac{3}{4}z^3p^2 \\ +\frac{1}{2}zp^2 \\ +\frac{35}{112}z^6p \\ +\frac{15}{40}z^4p \\ +\frac{1}{2}z^2p \\ p \\ +\frac{35}{1152}z^9 \\ +\frac{5}{112}z^7 \\ +\frac{3}{40}z^5 \\ +\frac{1}{6}z^3 \\ \cancel{+z} \\ \cancel{-z} \end{cases}$$

substitute

$$0 = \begin{cases} +\frac{1}{6}p^3 \\ +\frac{3}{4}z^3p^2 \\ +\frac{1}{2}zp^2 \\ +\frac{35}{112}z^6p \\ +\frac{15}{40}z^4p \\ +\frac{1}{2}z^2p \\ p \\ +\frac{35}{1152}z^9 \\ +\frac{5}{112}z^7 \\ +\frac{3}{40}z^5 \\ +\frac{1}{6}z^3 \\ \cancel{z} \\ \cancel{z} \end{cases}$$

$$p = -\frac{1}{6}z^3 + q$$

$$0 = \begin{cases} +\frac{1}{6}\left(-\frac{1}{6^3}z^9\right) + \dots & \frac{9-9}{2} = 0 \\ +\frac{3}{4}z^3\left(\frac{1}{36}z^6\right) + \dots & \frac{9-9}{2} = 0 \\ +\frac{1}{2}z\left(\frac{1}{36}z^6 - \frac{1}{3}qz^3 + \dots\right) & \frac{9-7}{2} = 1 \\ +\frac{35}{112}z^6\left(-\frac{1}{6}z^3\right) + \dots & \frac{9-9}{2} = 0 \\ +\frac{15}{40}z^4\left(-\frac{1}{6}z^3 + q\right) & \frac{9-7}{2} = 1 \\ +\frac{1}{2}z^2\left(-\frac{1}{6}z^3 + q\right) & \frac{9-5}{2} = 2 \\ -\frac{1}{6}z^3 + q \\ +\frac{35}{1152}z^9 \\ +\frac{5}{112}z^7 \\ +\frac{3}{40}z^5 \\ +\frac{1}{6}z^3 \\ +z \\ -z \end{cases}$$

add like terms...

$$0 = \left\{ \begin{array}{l} \left(-\frac{1}{1296} + \frac{1}{48} - \frac{35}{672} + \frac{35}{1152}\right)z^9 \\ \left(+\frac{1}{72} - \frac{15}{240} + \frac{5}{112}\right)z^7 \\ \left(+\frac{15}{40} - \frac{1}{6}\right)qz^4 \\ \left(-\frac{1}{12} + \frac{3}{40}\right)z^5 \\ +\frac{1}{2}z^2q \\ \cancel{-\frac{1}{6}z^3} + q \\ \cancel{+\frac{1}{6}z^3} \end{array} \right.$$

$$0 = \left\{ \begin{array}{l} \left(-\frac{1}{1296} + \frac{1}{48} - \frac{35}{672} + \frac{35}{1152}\right)z^9 \\ \left(+\frac{1}{72} - \frac{15}{240} + \frac{5}{112}\right)z^7 \\ \left(+\frac{15}{40} - \frac{1}{6}\right)qz^4 \\ \left(-\frac{1}{12} + \frac{3}{40}\right)z^5 \\ +\frac{1}{2}z^2q \\ -\frac{1}{6}z^3 + q \\ +\frac{1}{6}z^3 \end{array} \right.$$

Linear in q !

$$\frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 = q\left(1 + \frac{1}{2}z^2 + \frac{5}{24}z^4\right)$$

$$\frac{\frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9}{1 + \frac{1}{2}z^2 + \frac{5}{24}z^4} = q$$

Long Division!

$$\begin{aligned} & \left(1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 \right) \left(\frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 \right) \left(\frac{1}{120}z^5 + \dots \right) \\ & \quad \swarrow \quad \searrow \\ & \quad \quad \quad \% \end{aligned}$$

$$\begin{array}{r}
 \left(1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 \right) \left(\frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 \right) - \frac{1}{120}z^5 \\
 \hline
 \frac{1}{120}z^5 - \frac{1}{240}z^7 - \frac{5}{2880}z^9 \\
 \hline
 0 - \frac{1}{5040}z^7 - \frac{1}{10368}z^9
 \end{array}$$

X

$$\begin{array}{r}
 \textcircled{1} + \frac{1}{2}z^2 + \frac{5}{24}z^4 \quad / \quad \frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 \quad / \quad \frac{1}{120}z^5 - \textcircled{\frac{1}{5040}z^7} \\
 - \frac{1}{120}z^5 - \frac{1}{240}z^7 - \frac{5}{2880}z^9 + \\
 \hline
 0 \quad \textcircled{-\frac{1}{5040}z^7} - \frac{1}{10368}z^9
 \end{array}$$

An arrow points from the circled 1 to the circled $-\frac{1}{5040}z^7$. A percentage symbol (%) is written near the arrow.

X

$$\begin{array}{r}
 \left(1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 \right) \left(\frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 \right) - \frac{1}{120}z^5 - \frac{1}{5040}z^7 \\
 - \frac{1}{120}z^5 - \frac{1}{240}z^7 - \frac{5}{2880}z^9 + \\
 \hline
 0 - \frac{1}{5040}z^7 - \frac{1}{10368}z^9 \\
 + \frac{1}{5040}z^7 + \frac{1}{10080}z^9 \\
 \hline
 0 \quad \frac{1}{362880}z^9
 \end{array}$$

$$\begin{array}{r}
 \textcircled{1} + \frac{1}{2}z^2 + \frac{5}{24}z^4 \quad / \quad \frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 \quad \setminus \quad \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \textcircled{\frac{1}{362880}z^9} \\
 \hline
 - \frac{1}{120}z^5 - \frac{1}{240}z^7 - \frac{5}{2880}z^9 + \\
 \hline
 0 \quad - \frac{1}{5040}z^7 - \frac{1}{10368}z^9 \\
 + \frac{1}{5040}z^7 + \frac{1}{10080}z^9 \\
 \hline
 0 \quad \textcircled{\frac{1}{362880}z^9}
 \end{array}$$

A blue arrow points from the circled $\frac{1}{362880}z^9$ term in the final result back to the circled constant 1 in the first term of the dividend. A blue percentage sign (%) is placed near the arrow.

Long division:

$$q = \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$$

$$x = z - \frac{1}{6}z^3 + q$$

$$x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 + O(z^{11})$$

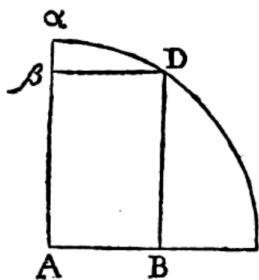
Long division:

$$q = \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$$

$$x = z - \frac{1}{6}z^3 + q$$

$$x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 + O(z^{11})$$

$x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$		
$x = z + p$	$ \begin{aligned} & + \frac{35}{1152}x^9 \\ & + \frac{5}{112}x^7 \\ & + \frac{3}{40}x^5 \\ & + \frac{1}{6}x^3 \\ & + x \\ & - z \end{aligned} $	$ \begin{aligned} & + \frac{35}{1152}z^9 + \dots \\ & + \frac{5}{112}(z^7 + 7z^6p) + \dots \\ & + \frac{3}{40}(z^5 + 5z^4p + 10z^3p^2) + \dots \\ & + \frac{1}{6}(z^3 + 3z^2p + 3zp^2 + p^3) \\ & + z + p \\ & - z \end{aligned} $
$p = -\frac{1}{6}z^3 + q$	$ \begin{aligned} & + \frac{1}{6}p^3 \\ & + \frac{3}{4}z^3p^2 \\ & + \frac{1}{2}zp^2 \\ & + \frac{35}{112}z^6p \\ & + \frac{15}{40}z^4p \\ & + \frac{1}{2}z^2p \\ & p \\ & + \frac{35}{1152}z^9 \\ & + \frac{5}{112}z^7 \\ & + \frac{3}{40}z^5 \\ & + \frac{1}{6}z^3 \\ & + z \\ & - z \end{aligned} $	$ \begin{aligned} & - \frac{1}{1296}z^9 + \dots \\ & + \frac{1}{48}z^9 + \dots \\ & + \frac{1}{72}z^7 - \frac{1}{6}qz^4 + \dots \\ & - \frac{35}{672}z^9 + \dots \\ & - \frac{15}{240}z^7 + \frac{15}{40}qz^4 + \dots \\ & - \frac{1}{12}z^5 + \frac{1}{2}z^2q \\ & - \frac{1}{6}z^3 + q \\ & + \frac{35}{1152}z^9 \\ & + \frac{5}{112}z^7 \\ & + \frac{3}{40}z^5 \\ & + \frac{1}{6}z^3 \\ & + z \\ & - z \end{aligned} $
$1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 \quad \left \quad \frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 \quad \right \quad \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$		



45. If from the Arch αD given the Sine AB was required; I extract the Root of the Equation found above, *viz.* $x = x + \frac{1}{6}x^3 + \frac{1}{40}x^5 + \frac{1}{112}x^7$ (it being supposed that $AB = x$, $\alpha D = x$, and $A\alpha = 1$) by which I find $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{161280}z^9$ &c. (1711)

$$x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

(2026)

**Finding $\cos(z)$ By Applying the
Binomial Series To $\sqrt{1 - \sin^2(z)}$**

Contents

1. Finding a Power Series For $z = \sin^{-1}(x)$
2. Inverting $z = \sin^{-1}(x)$ to find $x = \sin(z)$
3. Finding $\cos(z)$ By Applying the Binomial Series To $\sqrt{1 - \sin^2(z)}$

Questions?

Newton, Isaac, and John Stewart. *Sir Isaac Newton's Two Treatises of the Quadrature of Curves, And Analysis by Equations of an Infinite Number of Terms Explained*. Printed by James Bettenham for J. Nourse, 1745. https://books.google.it/books?id=noQ_AAAAcAAJ.