

# **Presentation: Newton series inversion**

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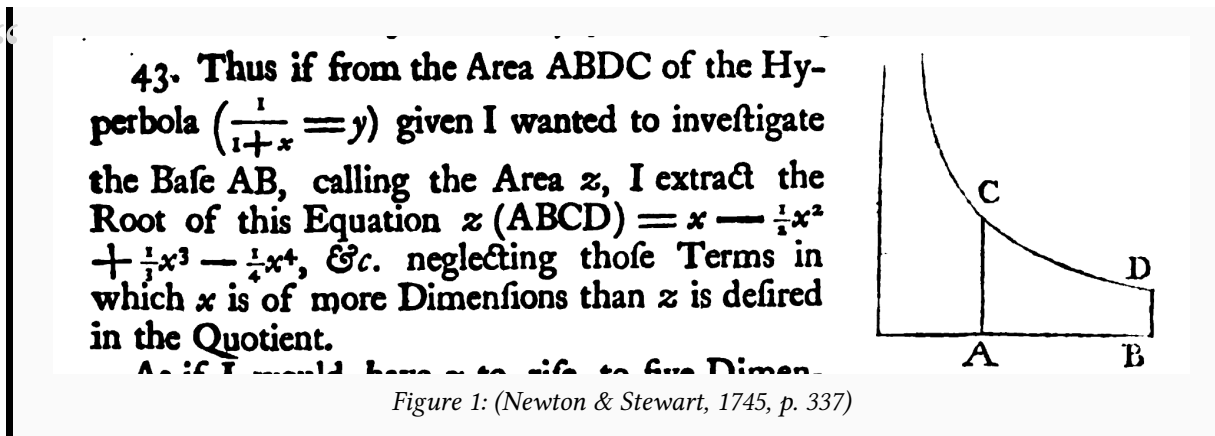
## 1 Intro

Now that Newton has found the series for arcsine (Equation 1), he inverts it to find the series expansion for the sine.

$$z(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \dots \tag{1}$$

## 2 Inverting $z = \ln(1 + x)$

He first demonstrates his series inversion method by inverting  $z = \ln(1 + x)$  to get  $x = e^z - 1$ :



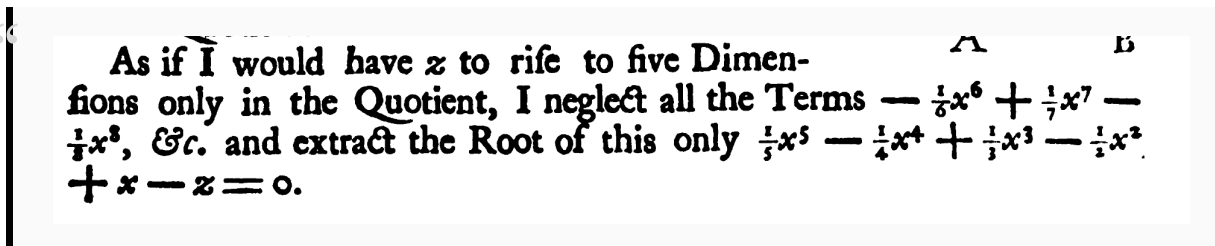
So Newton starts with the power series:

$$\ln(1 + x) \equiv z(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \dots \tag{2}$$

His aim is to find an expression like:

$$x = \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \dots \tag{3}$$

He is aware that if he wants to find coefficients up to  $z^5$  he only needs to consider the expansion of  $z(x)$  up to  $x^5$ :



So, the infinite series:

$$z(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \dots \quad (4)$$

and the finite series:

$$z_5(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 \quad (5)$$

... will have a different inverse in general, but their inverses will be identical for the terms up to  $z^5$ . Therefore, Newton can focus on inverting just the finite series  $z_5(x)$ . The inverses  $z_{n(x)}$  and  $x_{n(z)}$  are plotted for the first 12 have been plotted in an applet<sup>1</sup>

Newton next rewrites Equation 5 and puts all terms on the LHS:

$$-z + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 = 0 \quad (6)$$

We see clearly from the expansion that:

$$z(x) = x + O(x^2) \quad (7)$$

Therefore, the first-order approximation for  $z$ , which we call  $z_1(x)$ , is:

$$z_1(x) = x \quad (8)$$

The first-order approximation for  $x(z)$  is given by inverting the first-order approximation for  $z(x)$ . In other words,

$$x_1(z) = z \quad (9)$$

or

$$x = z \quad \text{to first order} \quad (10)$$

Now, Newton rewrites Equation 6 by giving each power of  $x$  a separate row:

$$0 = \begin{cases} +\frac{1}{5}x^5 \\ -\frac{1}{4}x^4 \\ +\frac{1}{3}x^3 \\ -\frac{1}{2}x^2 \\ + x \\ - z \end{cases} \quad (11)$$

Now we make our substitution  $x = z + p$ , where  $p$  is taken to be  $O(z^2)$

$$0 = \begin{cases} +\frac{1}{5}(z+p)^5 \\ -\frac{1}{4}(z+p)^4 \\ +\frac{1}{3}(z+p)^3 \\ -\frac{1}{2}(z+p)^2 \\ + z + p \\ - z \end{cases} \quad (12)$$

Then he expands Equation 12 selectively as follows:

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<sup>1</sup>(Elgersma, 2026)

$$0 = \begin{cases} +\frac{1}{5}z^5 + z^4p + \dots & 5 - 5 = 0 \\ -\frac{1}{4}z^4 - z^3p + \frac{3}{2}z^2p^2 + \dots & 5 - 4 = 1 \\ +\frac{1}{3}z^3 + z^2p + zp^2 + \frac{1}{3}zp^3 + \dots & 5 - 3 = 2 \\ -\frac{1}{2}z^2 - zp - \frac{1}{2}p^2 + 0 & 5 - 2 = 3 \\ \cancel{+z^1 + p} \\ \cancel{z^1} \end{cases} \quad (13)$$

We have kept the first ignored terms in grey, and point out that they are all of a higher order in  $z$  than  $z^5$ . Newton uses the following rule to determine which terms to keep:

**Rule; That after the first Term resulting from each Quantity that is collateral to it, I add no more Terms upon the right Hand than the Index of the Dimension of that first Term wants Units of the Index of the greatest Dimension. As in this Example, where the greatest Dimension is 5, I neglect all the Terms after  $z^5$ , I put one after  $z^4$ , and two only after  $z^3$ . When the Root ( $x$ ) to be extracted,**

The *Units of the Index of the greatest dimension* is 5. The *Index of the Dimension of the First term* is 5 for the top row, and decreases by one for each row. Therefore  $5 - 5 = 0$  and *no* additional terms are added after the  $z^5$  term. The same logic leads us to conclude that one term is added after the  $z^4$  term, and two after the  $z^3$  term.

This means that terms like  $z^4p$  or  $z^2p^2$ , which are  $\sim O(z^6)$ , will be dropped. These terms are irrelevant because we are only seeking to find the coefficients up to  $z^5$ .

We are thus left with the following expression:

$$0 = \begin{cases} +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 - z^3p \\ +\frac{1}{3}z^3 + z^2p + zp^2 \\ -\frac{1}{2}z^2 - zp - \frac{1}{2}p^2 \\ +p \end{cases} \quad (14)$$

Or, in Newton's original text:

$$\begin{array}{r}
+ \frac{1}{5}z^5 \text{ } \mathcal{O}c. \\
- \frac{1}{4}z^4 - z^3p \text{ } \mathcal{O}c. \\
+ \frac{1}{3}z^3 + z^2p + zp^2 \text{ } \mathcal{O}c. \\
- \frac{1}{2}z^2 - zp - \frac{1}{2}p^2 \\
+ z + p \\
- z
\end{array}$$

Figure 2: (Newton & Stewart, 1745, p. 337)

We remember that we are looking for the lowest term in the power series expansion of  $p(z)$  (where  $p$  is the correction to the current approximation  $x = z$ ). Since we assume  $p \sim O(z^2)$ , we can ignore terms like  $p^2$ . Collecting like terms:

$$\begin{aligned}
0 = -\frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \frac{1}{5}z^5 \\
+ p(1 - z + z^2 - z^3)
\end{aligned} \tag{15}$$

Whence:

$$p = \frac{+\frac{1}{2}z^2 - \frac{1}{3}z^3 + \frac{1}{4}z^4 - \frac{1}{5}z^5}{1 - z + z^2 - z^3} \tag{16}$$

Since we only seek the lowest-order coefficient of  $p$ , it is enough to expand  $(1 - z + z^2 - z^3)^{-1} = (1 + O(z))$ , so

$$p = \left(\frac{1}{2}z^2 - \frac{1}{3}z^3 + \frac{1}{4}z^4 - \frac{1}{5}z^5\right)(1 + O(z)) = \frac{1}{2}z^2 + O(z^3) \tag{17}$$

We can expand the denominator to first order with the binomial theorem to find that  $p(z) = \frac{1}{2}z^2 + O(z^3) + \dots$

So we have now found our second order expansion of  $x(z)$ :

$$x_2(z) = z + \frac{z^2}{2} \tag{18}$$

Newton's table up until this point looks like this:

$z + p = x$	$ \begin{array}{r} + \frac{1}{5}x^5 \\ - \frac{1}{4}x^4 \\ + \frac{1}{3}x^3 \\ - \frac{1}{2}x^2 \\ + x \\ - z \end{array} $	$ \begin{array}{r} + \frac{1}{5}z^5 \text{ & } \mathcal{O}c. \\ - \frac{1}{4}z^4 - z^3p \text{ & } \mathcal{O}c. \\ + \frac{1}{3}z^3 + z^2p + zp^2 \text{ & } \mathcal{O}c. \\ - \frac{1}{2}z^2 - zp - \frac{1}{2}p^2 \\ + z + p \\ - z \end{array} $
$\frac{1}{2}z^2 + q = p$		

Figure 3: The steps to find the first and second terms in the inversion of  $\ln(1+x)$

Now Newton returns to Equation 14, and rewrites it in order of decreasing dimensions of  $p$  and then  $z$ :

$$0 = \begin{cases} zp^2 \\ -\frac{1}{2}p^2 \\ -z^3p \\ +z^2p \\ -zp \\ +p \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases} \quad (19)$$

In Newton's original text, this looks like this:

$x + p = x$	$ \begin{array}{r} + \frac{1}{5}x^5 \\ - \frac{1}{4}x^4 \\ + \frac{1}{3}x^3 \\ - \frac{1}{2}x^2 \\ + x \\ - x \end{array} $	$ \begin{array}{r} + \frac{1}{5}z^5 \text{ \&Ocirc;} \\ - \frac{1}{4}z^4 - z^3p \text{ \&Ocirc;} \\ + \frac{1}{3}z^3 + z^2p + zp^2 \text{ \&Ocirc;} \\ - \frac{1}{2}z^2 - zp - \frac{1}{2}p^2 \\ + z + p \\ - z \end{array} $
$\frac{1}{2}z^2 + q = p$	$ \begin{array}{r} + zp^2 \\ - \frac{1}{2}p^2 \\ - z^3p \\ + z^2p \\ - zp \\ + p \\ + \frac{1}{5}z^5 \\ - \frac{1}{4}z^4 \\ + \frac{1}{3}z^3 \\ - \frac{1}{2}z^2 \end{array} $	

Now, he substitutes  $p = \frac{1}{2}z^2 + q$ , where  $q$  is assumed to be of the form  $q = O(z^3) + \dots$ . He uses the same rules to determine which terms to keep

$$0 = \begin{cases} +z(\frac{1}{2}z^2 + q)^2 \\ -\frac{1}{2}(\frac{1}{2}z^2 + q)^2 \\ -z^3(\frac{1}{2}z^2 + q) \\ +z^2(\frac{1}{2}z^2 + q) \\ -z(\frac{1}{2}z^2 + q) \\ +(\frac{1}{2}z^2 + q) \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases} = \begin{cases} \frac{1}{4}z^5 + z^3q \\ -\frac{1}{8}z^4 - \frac{1}{2}z^2q \\ -\frac{1}{2}z^5 - z^3q \\ +\frac{1}{2}z^4 + z^2q \\ -\frac{1}{2}z^3 - qz \\ +\frac{1}{2}z^2 + q \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases} \tag{20}$$

At this point, there are no quadratic terms in  $q$ . This makes our situation a little bit different from before. We no longer need to explicitly reject terms of order  $q^2$  like we did for  $p^2$ . Instead, we have a closed-form solution for  $q$  in terms of  $z$ , valid for terms up to  $z^5$ :

$$x = z + \frac{1}{2}z^2 + q \tag{21}$$

where  $q$  is found by solving for  $q$  in Equation 20:

$$q = \frac{+\frac{1}{20}z^5 - \frac{1}{8}z^4 + \frac{1}{6}z^3}{1 - z + \frac{1}{2}z^2} \tag{22}$$

This can be found by straightforward polynomial long division, as Newton explains in §44.2:

2. When I see that  $p$ ,  $q$ , or  $r$ , &c. in the last resulting Equation, is found of one Dimension only, I seek it's Value, that is to say the remaining Terms, which are still to be added to the Quotient, by means of Division; as you see done here.

the steps are detailed below:

$$\begin{array}{r} 1 \quad -z + \frac{1}{2}z^2 \quad \Big/ \quad \frac{1}{6}z^3 \\ \underline{\phantom{1} \phantom{-z} \phantom{\frac{1}{2}z^2} \phantom{\frac{1}{6}z^3}} \\ \phantom{1} \phantom{-z} \phantom{\frac{1}{2}z^2} \phantom{\frac{1}{6}z^3} \end{array}$$

Figure 5: Step 1: Dividing  $\frac{1}{6}z^3$  by 1

$$\begin{array}{r} 1 \quad -z + \frac{1}{2}z^2 \quad \Big/ \quad \frac{1}{6}z^3 + \frac{-1}{8}z^4 + \frac{1}{20}z^5 \\ \underline{-\frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5} \\ 0 \quad + \frac{1}{24}z^4 + \frac{1}{30}z^5 \end{array}$$

Figure 6: Step 2: Calculating the remainder

$$\begin{array}{r} 1 \quad -z + \frac{1}{2}z^2 \quad \Big/ \quad \frac{1}{6}z^3 + \frac{-1}{8}z^4 + \frac{1}{20}z^5 \\ \underline{-\frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5} \\ 0 \quad + \frac{1}{24}z^4 + \frac{1}{30}z^5 \end{array}$$

Figure 7: Step 3: Dividing  $\frac{1}{24}z^4$  by 1

$$\begin{array}{r} 1 \quad -z + \frac{1}{2}z^2 \quad \Big/ \quad \frac{1}{6}z^3 + \frac{-1}{8}z^4 + \frac{1}{20}z^5 \\ \underline{-\frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5} \\ 0 \quad + \frac{1}{24}z^4 + \frac{1}{30}z^5 \\ \underline{-\frac{1}{24}z^4 + \frac{1}{24}z^5} \\ 0 \quad + \frac{1}{120}z^5 \end{array}$$

Figure 8: Step 4: Calculating the remainder

$$\begin{array}{r} 1 \quad -z + \frac{1}{2}z^2 \quad \Big/ \quad \frac{1}{6}z^3 + \frac{-1}{8}z^4 + \frac{1}{20}z^5 \\ \underline{-\frac{1}{6}z^3 + \frac{1}{6}z^4 - \frac{1}{12}z^5} \\ 0 \quad + \frac{1}{24}z^4 + \frac{1}{30}z^5 \\ \underline{-\frac{1}{24}z^4 + \frac{1}{24}z^5} \\ 0 \quad + \frac{1}{120}z^5 \end{array}$$

Figure 9: Step 5: Dividing  $\frac{1}{120}z^5$  by 1 to yield the final term

At this point we can stop, remembering that we sought the coefficients up to  $z^5$ . Thus:

$$x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 + \dots \quad (23)$$

If we wanted to find the next coefficient, we would have had to start with one more power of  $x$  in the series for  $z$ , namely:

$$z_6(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 \quad (24)$$

And use the same method.

So, William Dunham was exaggerating when he rhetorically asks why we can trust the answer when “we are discarding most terms most of the time” (Dunham, 1991, p. 11). No, we are only discarding terms that are higher powers than the one we are calculating at each step. The method is not spurious.

This method is closely related to the modern method of undetermined coefficients, except the latter makes it more explicit that we are assuming an equation of a certain form, whereas Newton’s method leaves that implicit. Nevertheless, it is clear from the assumptions that he makes about which terms to leave out, that he expects a series of the form of Equation 3.

### 3 Inverting $z = \sin^{-1}(x)$ to get $x = \sin(z)$ .

Now that we have learned Newton’s method with a simpler example, let’s apply it to inverting the series for arcsine, namely:

$$z(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \dots \quad (25)$$

This series is simpler because all the coefficients are positive, and there are only odd terms. The coefficients themselves are more complicated fractions, however. This time we want a solution up to the power of  $z^9$ .

The first order expansion will be:

$$z_1(x) = x \quad (26)$$

So we find the next order by substituting  $z = x + p$  into Equation 25. Following Newton’s example, we start by truncating our infinite series and moving all the terms to the RHS:

$$0 = \begin{cases} +\frac{35}{1152}x^9 \\ +\frac{5}{112}x^7 \\ +\frac{3}{40}x^5 \\ +\frac{1}{6}x^3 \\ +x \\ -z \end{cases} \quad (27)$$

Now we substitute the expansion  $x = z + p$

$$0 = \begin{cases} +\frac{35}{1152}(z+p)^9 \\ +\frac{5}{112}(z+p)^7 \\ +\frac{3}{40}(z+p)^5 \\ +\frac{1}{6}(z+p)^3 \\ +x \\ -z \end{cases} \quad (28)$$

Again, we don’t need to expand all the terms, only the ones that will be of a lower order than  $x^9$ , which we have done below.

$$0 = \begin{cases} +\frac{35}{1152}(z^9) + \dots & \frac{9-9}{2} = 0 \\ +\frac{5}{112}(z^7 + 7z^6p) + \dots & \frac{9-7}{2} = 1 \\ +\frac{3}{40}(z^5 + 5z^4p + 10z^3p^2) + \dots & \frac{9-5}{2} = 2 \\ +\frac{1}{6}(z^3 + 3z^2p + 3zp^2 + p^3) + \dots & \frac{9-3}{2} = 3 \\ +z + p \\ -z \end{cases} \quad (29)$$

This step relies on the assumption that the lowest term in  $p$  will be  $O(x^3)^2$ . In fact, we expect only odd powers to appear in the power series for  $x(z)$  because the original series contains only odd powers and the inverse of an odd function is odd. In other words, it is assumed from the start that the power series will only have odd coefficients - which is the same assumption we make in the modern ‘method of undetermined coefficients’.

Newton explains which terms to keep as follows:

after  $z^4$ , and two only after  $z^3$ . When the Root ( $x$ ) to be extracted, is every where of even or odd Dimensions, let this be the Rule: That after the first Term, resulting from each Quantity which is collateral to it, you add no more Terms towards the Right Hand, than what the Index of the Dimension of that first Term, wants Pairs of Units of the Index of the highest Dimension; or no more than what it wants Ternaries of Units, when the Indexes of the Dimensions of  $x$  differ by three Units; and so in others.

The “Index of the Highest Dimension” is 9. The “index of the dimension of the first term” is 9 for the first row and drops by 2 in each row. “wants Pairs of Units” means we have to divide the difference by two to get the number of additional units on the right hand side.

We now want to find the lowest order of  $p$ , so we take Equation 29, drop all higher powers of  $p$ , and solve for  $p$

$$0 = p \left( \frac{35}{112}z^6 + \frac{5 \times 3}{40}z^4 + \frac{3}{6}z^2 + 1 \right) + \frac{35}{1152}z^9 + \frac{5}{112}z^7 + \frac{3}{40}z^5 + \frac{1}{6}z^3 \quad (30)$$

$$p = \frac{-\frac{1}{6}z^3 - \frac{3}{40}z^5 - \frac{5}{112}z^7 - \frac{35}{1152}z^9}{1 + \frac{1}{2}z^2 + \frac{5 \times 3}{40}z^4 + \frac{7 \times 5}{112}z^6} \quad (31)$$

We can see that the lowest order term is  $p(z) = -\frac{1}{6}z^3$ . We have thus found the third-order approximation.

$$x_3(z) = z - \frac{1}{6}z^3 + \dots \quad (32)$$

We now rewrite Equation 29 in terms of decreasing powers of  $p$  and  $z$ , just like we did with Equation 19 when calculating the inverse of  $\ln(1+x)$  in Section 2. This arrangement makes it easier

<sup>2</sup>Consider the third row. If  $p$  had terms of  $O(x^2)$  then we would have to consider terms of order  $p^4$

to substitute successive approximations for  $p$ , since terms of the same degree in  $p$  are grouped together exactly as in Newton's original tables.

$$0 = \left\{ \begin{array}{l} +\frac{1}{6}p^3 \\ +\frac{3}{4}z^3p^2 \\ +\frac{1}{2}zp^2 \\ +\frac{35}{112}z^6p \\ +\frac{15}{40}z^4p \\ +\frac{1}{2}z^2p \\ p \\ +\frac{35}{1152}z^9 \\ +\frac{5}{112}z^7 \\ +\frac{3}{40}z^5 \\ +\frac{1}{6}z^3 \\ \cancel{+z} \\ \cancel{-z} \end{array} \right. \quad (33)$$

We now use the substitution  $p = -\frac{1}{6}z^3 + q(z)$  into Equation 33 where we assume  $q(z) \sim O(z^5)$ . Again, we use Newton's rule for how many terms to keep on the RHS.

$$0 = \left\{ \begin{array}{l} +\frac{1}{6}\left(-\frac{1}{6^3}z^9\right) + \dots \quad \frac{9-9}{2} = 0 \\ +\frac{3}{4}z^3\left(\frac{1}{36}z^6\right) + \dots \quad \frac{9-9}{2} = 0 \\ +\frac{1}{2}z\left(\frac{1}{36}z^6 - \frac{1}{3}qz^3 + \dots\right) \quad \frac{9-7}{2} = 1 \\ +\frac{35}{112}z^6\left(-\frac{1}{6}z^3\right) + \dots \quad \frac{9-9}{2} = 0 \\ +\frac{15}{40}z^4\left(-\frac{1}{6}z^3 + q\right) \quad \frac{9-7}{2} = 1 \\ +\frac{1}{2}z^2\left(-\frac{1}{6}z^3 + q\right) \quad \frac{9-5}{2} = 2 \\ -\frac{1}{6}z^3 + q \\ +\frac{35}{1152}z^9 \\ +\frac{5}{112}z^7 \\ +\frac{3}{40}z^5 \\ +\frac{1}{6}z^3 \\ +z \\ -z \end{array} \right. \quad (34)$$

We can expand the brackets and add like terms:

$$0 = \left\{ \begin{array}{l} \left(-\frac{1}{1296} + \frac{1}{48} - \frac{35}{672} + \frac{35}{1152}\right)z^9 \\ \left(+\frac{1}{72} - \frac{15}{240} + \frac{5}{112}\right)z^7 \\ \left(+\frac{15}{40} - \frac{1}{6}\right)qz^4 \\ \left(-\frac{1}{12} + \frac{3}{40}\right)z^5 \\ +\frac{1}{2}z^2q \\ \cancel{-\frac{1}{6}z^3} + q \\ \cancel{+\frac{1}{6}z^3} \end{array} \right. \quad (35)$$

At this point we find that our equation is linear in  $q$ , which means we can solve it using straightforward polynomial division:

$$0 = \begin{cases} -\frac{17}{10368}z^9 \\ -\frac{1}{252}z^7 \\ +\frac{5}{24}qz^4 \\ -\frac{1}{120}z^5 \\ +\frac{1}{2}z^2q \\ -\frac{1}{6}z^3 + q \\ +\frac{1}{6}z^3 \end{cases} \quad (36)$$

$$\frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9 = q \left( 1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 \right) \quad (37)$$

$$\frac{\frac{1}{120}z^5 + \frac{1}{252}z^7 + \frac{17}{10368}z^9}{1 + \frac{1}{2}z^2 + \frac{5}{24}z^4} = q \quad (38)$$

We can calculate the remaining terms by straightforward polynomial division:

Figure 10: Step 1: Dividing  $\frac{1}{6}z^3$  by 1

Figure 11: Step 2: Calculating the remainder

Figure 12: Step 3: Dividing  $\frac{1}{24}z^4$  by 1

Figure 13: Step 4: Calculating the remainder

Figure 14: Step 4: Calculating the remainder

We found that

$$q = \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 \quad (39)$$

And therefore

$$x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 + O(z^{11}) \quad (40)$$

We recognize this as the first few terms of the formula:

$$x = \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} \quad (41)$$

To calculate the  $z^{13}$  we would have to start with the expansion of  $z(x)$  up to order  $x^{15}$

## Bibliography

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