Presentation: Newton series inversion

Contents

1 Inverting $\ln(1+x)$ to get $e^x - 1$

To explain his series inversion trick, he uses a simpler example, namely the inversion of $z = \ln(1+x)$.

In modern terms, he is finding the inverse function of $\ln(1+x)$, namely $e^x - 1$.

He starts with the power series:

$$\ln(1+x) = z(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \ldots \tag{1}$$

His aim is to find an expression like:

$$x = \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \ldots \tag{2}$$

1.1 Truncate target series

The first thing he does is to decide what the highest power of z is that we want in $x(z)$. Newton chooses z^5:

> As if I would have z to riſe to five Dimenſions only in the Quotient, I neglect all the Terms $-\frac{1}{6}x^6 + \frac{1}{7}x^7 - \frac{1}{8}x^8$, &c. and extract the Root of this only $\frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - z = 0$.

Clearly, the truncated series $z_5(x)$ and the full series $z(x)$ have a different inverse, but Newton knows that their inverses will be identical for the terms up to z^5

$$z_5(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 \tag{3}$$

$$z(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \ldots \tag{4}$$

Thus he sets out to invert the finite series $z_5(x)$ and truncate his result after working out the coefficient of z^5. He starts by rewriting Equation 3 in the following form:

$$-z + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 = 0 \tag{5}$$

We see clearly from the expansion that:

$$z(x) = x + O(x^2) \tag{6}$$

Therefore, the first-order approximation for z, which we call $z_1(x)$, is:

$$z_1(x) = x \tag{7}$$

The first-order approximation for $x(z)$ is given by inverting the first-order approximation for $z(x)$. In other words,

$$x_1(z) = z \tag{8}$$

We can find the second-order approximation for x, by inverting the second-order approximation $z_2 = x - \frac{1}{2}x^2$.

We substitute $x = z + p(z)$, where we assume $p(z) \sim O(z^2)$

$$\begin{aligned} z &= z + p - \frac{1}{2}(z+p)^2 \\ \not{z} &= \not{z} + p - \frac{1}{2}(z^2 + 2zp + p^2) \end{aligned} \tag{9}$$

We want to find the lowest coefficient of z in p. $p \sim O(z^2)$ allows us to neglect the term p^2 and zp, leaving us with:

$$p = \frac{1}{2}z^2 \tag{10}$$

In fact, Newton doesn't start with the first-order approximation for z and successively inverts higher and higher approximations. Instead he works on the fifth-order approximation directly. He first rewrites Equation 5 by giving each power of x a separate row:

$$0 = \begin{cases} +\frac{1}{5}x^5 \\ -\frac{1}{4}x^4 \\ +\frac{1}{3}x^3 \\ -\frac{1}{2}x^2 \\ + \ \ x \\ - \ \ z \end{cases} \tag{11}$$

Now we make our substitution $x = z + p(z)$

$$0 = \begin{cases} +\frac{1}{5}(z+p)^5 \\ -\frac{1}{4}(z+p)^4 \\ +\frac{1}{3}(z+p)^3 \\ -\frac{1}{2}(z+p)^2 \\ + \ \ z + p \\ - \ \ z \end{cases} \tag{12}$$

Then he expands Equation 12 selectively, according to the following rule:

> Rule; That after the firſt Term reſulting from each Quantity that is collateral to it, I add no more Terms upon the right Hand than the Index of the Dimenſion of that firſt Term wants Units of the Index of the greateſt Dimenſion. As in this Example, where the greateſt Dimenſion is 5, I neglect all the Terms after z^5, I put one after z^4, and two only after z^3. When the Root *(x)* to be extracted,

The “Units of the Index of the greatest dimension” is 5. The “Index of the Dimension of the First term” is 5 for the top row, and decreases by one for each row. Therefore $5-5=0$ and *no* additional terms are added after the z^5 term. The same logic leads us to conclude that one term is added after the z^4 term, and two after the z^3 term.

This means that terms like z^4p or z^2p^2, which are $\sim O(z^6)$, will be dropped. These terms are irrelevant because we are only seeking to find the coefficients up to z^5. I have added the next, ignored term in grey in each case:

$$0 = \begin{cases} +\frac{1}{5}z^5 + z^4p + \ldots & 5-5=0 \\ -\frac{1}{4}z^4 - z^3p + \frac{3}{2}z^2p^2 + \ldots. & 5-4=1 \\ +\frac{1}{3}z^3 + z^2p + zp^2 + \frac{1}{3}zp^3 + \ldots. & 5-3=2 \\ -\frac{1}{2}z^2 - zp - \frac{1}{2}p^2 + 0 & 5-2=3 \\ \quad \cancel{+z^1} + 0 + 0 + 0 + 0 & 5-1=4 \\ \quad \cancel{-z^1} + 0 + 0 + 0 + 0 + 0 & 5-0=5 \end{cases} \tag{13}$$

Now that we have motivated ignoring most terms after expanding the brackets, we are left with the following simple expression

$$0 = \begin{cases} +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 - z^3p \\ +\frac{1}{3}z^3 + z^2p + zp^2 \\ -\frac{1}{2}z^2 - zp - \frac{1}{2}p^2 \end{cases} \tag{14}$$

We remember that we are looking for the lowest term in the power series expansion of $p(z)$. Since we assume $p \sim O(z^2)$, we can ignore terms like p^2. Collecting like terms:

$$\begin{aligned} 0 = -\frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \frac{1}{5}z^5 \\ +p(1 - z + z^2 - z^3) \end{aligned} \tag{15}$$

Whence:

$$p = \frac{+\frac{1}{2}z^2 - \frac{1}{3}z^3 + \frac{1}{4}z^4 - \frac{1}{5}z^5}{1 - z + z^2 - z^3} \tag{16}$$

We can expand the denominator to first order with the binomial theorem to find that $p(z) = \frac{1}{2}z^2 + O(z^3) + \ldots$.

So we have now found our second order expansion of $x(z)$:

$$x_2(z) = z + \frac{z^2}{2} \tag{17}$$

Newton's table up until this point looks like this:

$z + p = x$	$+\frac{1}{5}x^5$ $-\frac{1}{4}x^4$ $+\frac{1}{3}x^3$ $-\frac{1}{2}x^2$ $+x$ $-z$	$+\frac{1}{5}z^5$ &c. $-\frac{1}{4}z^4 - z^3p$ &c. $+\frac{1}{3}z^3 + z^2p + zp^2$ &c. $-\frac{1}{2}z^2 - zp - \frac{1}{2}p^2$ $+z + p$ $-z$
$\frac{1}{2}z^2 + q = p$		

Figure 1: The steps to find the first and second terms in the inversion of $\ln(1 + x)$

Now Newton returns to Equation 14, and rewrites it in order of decreasing dimensions of p and then z:

$$0 = \begin{cases} zp^2 \\ -\frac{1}{2}p^2 \\ -z^3p \\ +z^2p \\ -zp \\ +p \\ +\frac{1}{5}z^5 \\ -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 \end{cases} \tag{18}$$

In Newton's original text, this looks like this:

$z+p=x$	$+\frac{1}{5}x^5$ $-\frac{1}{4}x^4$ $+\frac{1}{3}x^3$ $-\frac{1}{2}x^2$ $+x$ $-z$	$+\frac{1}{5}z^5$ &c. $-\frac{1}{4}z^4-z^3p$ &c. $+\frac{1}{3}z^3+z^2p+zp^2$ &c. $-\frac{1}{2}z^2-zp-\frac{1}{2}p^2$ $+z\ +p$ $-z$
$\frac{1}{2}z^2+q=p$	$+zp^2$ $-\frac{1}{2}p^3$ $-z^3p$ $+z^2p$ $-zp$ $+p$ $+\frac{1}{5}z^5$ $-\frac{1}{4}z^4$ $+\frac{1}{3}z^3$ $-\frac{1}{2}z^2$	

1.2 Then he uses polynomial division for the rest.

So he ends up with:

$$x = z + \frac{1}{2}z^2 + \frac{\frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5}{1 - z + \frac{1}{2}z^2} \tag{19}$$

Finding:

$$x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 + \ldots \tag{20}$$

Calculate this to the power of 5. Note: we cannot keep going after the 5th term because

So X is wrong when he says “we are discarding most terms most of the time”. No, we are only discarding the ones that are irrelevant to calculating the coefficient we are trying to get

This is exactly the same as the method of undetermined coefficients, except the latter makes it more explicit that we are assuming an equation of a certain form, whereas Newton’s method leaves that implicit.

Then he uses polynomial division for the rest